

The adjacent concurrence on infinitesimally unstable Betti syzygy classes

Takeshi Usa

Dept. of Math. Univ. of Hyogo *

Abstract

Even for an arithmetically D_2 non-singular projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{C}) = P$, when we consider a 1-st infinitesimal embedded deformation of X in P , some classes in the q -th Betti syzygy space (or Koszul homology group) $T_m^{1,q} \cong (\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)})_{(m)}$ of X in (polynomial) degree m may suddenly disappear. When we once find such an infinitesimally unstable q -th Betti syzygy class, we often find a non-zero $(q+1)$ -th Betti syzygy space $T_m^{1,q+1}$ or a non-zero $(q-1)$ -th Betti syzygy space $T_m^{1,q-1}$ with the same polynomial degree m at the same time, which has also an infinitesimally unstable Betti syzygy class. One of the approximate explanations for such phenomena can be seen in the proof of Corollary 2.9 of [8], where we used a decomposition of the obstruction map $\delta_{IDF}^{(1)} = (ob_\sigma)_m^{1,q}$ (cf. Theorem 2.6 of [7]). Here, in a general situation, we give another but steadier explanation of these phenomena (cf. Main Theorem 1.16 (1.16.1)). As a byproduct, for a local 1-parameter family of canonical curves of genus 5 with a trigonal curve in the central fiber, we can determine the module structure of $\overline{\mathcal{F}}_3^{1,2}$ and the cohomological base change map $\overline{\varphi}_3^{1,2}$, which was a remaining problem of [9] and of [10].

Keywords: adjacent concurrence, degeneration of syzygies, canonical curve, trigonal curve

§0 Introduction.

For a closed subscheme $X \subseteq \mathbb{P}^N(\mathbb{C}) = Proj(S) = P$, let us consider the graded Betti numbers $\{\beta_{q,m}(X)\}$ of the homogeneous coordinate ring R_X of X , where $\beta_{q,m}(X) = \dim Tor_q^S(R_X, S/S_+)_{(m)}$. Over an algebraic scheme B , when we deform the scheme X in a projective and flat family $f: \mathfrak{X} \rightarrow B$ of closed subschemes in P , contrary to our naive expectation, the graded Betti numbers do *not* satisfy the upper semi-continuity on B in general. This is caused by the reason that even in the case of the embedded deformation, the family of closed subschemes does not induce a family of the (valid) homogeneous coordinate rings of the members (= the fibers). In other words, the associated S_+ -primary ideals of the homogeneous coordinate rings of the members in the family can not be removed simultaneously.

*2167 Shosha, Himeji, 671-2201 Japan.

E-mail address : usa@sci.u-hyogo.ac.jp

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If we assume moreover that all the closed fibers $X(b) = f^{-1}(b)$ ($b \in B^{(cl.)}$) of the family $f : \mathfrak{X} \rightarrow B$ satisfy the arithmetic D_2 -condition, then we have isomorphisms $T_m^{1,q}(b) := H^1(\Omega_P^q(m) \otimes I_{X(b)}) \cong \text{Tor}_q^S(R_{X(b)}, S/S_+)_m$ and see that the graded Betti numbers barely have the upper semi-continuity.

Let us consider the constancy of the graded Betti numbers which is a stronger property than the upper semi-continuity. Even if we suppose that the family $f : \mathfrak{X} \rightarrow B$ is smooth with adding to the assumption above, still the constancy of the graded Betti numbers does not hold in general. The decay of the constancy of the graded Betti numbers is named *the degeneration of syzygies*, which means that the limit of the systems of minimal generators of the syzygy modules loses its power of generation. One of the classical examples for this phenomenon can be found in [4] (cf. also [5], [6]), which is the family of canonical curves of genus 5 including trigonal curves as the special fibers. In our previous papers [9] and [10], we studied this example by using our technique of infinitesimal study of syzygies. This technique was prepared in [7], [8] and [11]. The original purpose of these three papers was studying the constancy of the graded Betti numbers by using the locally freeness of the Betti syzygy sheaves $\{\mathcal{T}_m^{1,q}(\mathfrak{X}) = R^1\pi_*(\Omega_{P \times B/B}^q(m) \otimes I_{\mathfrak{X}})|q, m \in \mathbb{Z}\}$, where $\pi : P \times B \rightarrow B$ denotes the projection morphism to B . Someone might think that we can study also the degeneration of syzygies by analyzing locally the non-freeness of the stalks of these Betti syzygy sheaves. However, without freeness of the stalks of the Betti syzygy sheaves, for a closed point $b \in B$, the cohomological base change map $\varphi_m^{1,q}(b) : \mathcal{T}_m^{1,q} \otimes k(b) \rightarrow T_m^{1,q}(X(b))$ is not surjective in general. The ruin of this cohomological base change property brings us a technical difficulty for studying the degeneration of syzygies by analyzing Betti syzygy sheaves in general. Although we have this technical difficulty, if we restrict ourselves to the minimal syzygy level of the degeneration, we still have a spy hole which make us possible to study the degeneration of syzygies.

In this paper, we turn this technical difficulty into our advantage for studying the degeneration of syzygies and find a general principle which is called as “adjacent concurrence” (cf. Main Theorem 1.16). Roughly speaking, this principle tells us that once we have a q -th Betti syzygy class in (polynomial) degree m which can not be extended to the (1-st) infinitesimal neighborhood in the direction of a global normal vector field $\sigma \in H^0(N_X)$, then we have also a $(q+1)$ -th Betti syzygy class or a $(q-1)$ -th Betti syzygy class in degree m with the similar property.

We refer fundamentally to [9], [10], [3] or [2], and often use the terminology and the results in [9], [10] or in [3] without mentioning except somethings important.

§1 Main results.

In this section, we start from a general setting, where we once leave the concrete example handled continuously in [9] and [10]. From [9], we recall Circumstances 1.1 (AD2) and Notation and Conventions 1.2 as this general setting.

Definition 1.1 *We call a family $f : \mathfrak{X} \rightarrow B$ satisfying the property in the circumstances (AD2) of [9] as an AD2-family in $\mathbb{P}^N(\mathbb{C}) = P$ for short.*

The induced sheaf $\mathcal{T}_m^{p,q} = \mathcal{T}_m^{p,q}(\mathfrak{X})$, the induced $O_{\overline{B}_\nu}$ -module $\overline{\mathcal{T}}_{m,\nu}^{p,q} = \overline{\mathcal{T}}_{m,\nu}^{p,q}(b_0)$ and the induced $k(b_0)$ -vector space $T_m^{p,q} = T_m^{p,q}(b_0)$ from the AD2-family $f : \mathfrak{X} \rightarrow B$ and a closed point $b_0 \in B$, are

called (p, q, m) -Betti syzygy sheaf (or simply, Betti syzygy sheaf), ν -th infinitesimal (p, q, m) -Betti syzygy module (or, infinitesimal Betti syzygy module) at the point b_0 , and (p, q, m) -Betti syzygy space at the point b_0 (or Betti syzygy space), respectively. An element of $T_m^{p,q}$ is called a $((p, q, m)$ -)Betti syzygy class.

The indexes p, q, m, ν of the infinitesimal Betti syzygy module $\overline{\mathcal{F}}_{m,\nu}^{p,q} = R^p \pi_\nu^*(\Omega_{P \times \overline{B}_\nu / \overline{B}_\nu}^q(m) \otimes I_{\overline{\mathfrak{X}}_\nu})$ are named as follows. We say that the infinitesimal Betti syzygy module $\overline{\mathcal{F}}_{m,\nu}^{p,q}$ has cohomology degree p , syzygy level q , polynomial degree (or S -degree) m , infinitesimal level ν . Also the indexes p, q, m of the Betti syzygy sheaf $\mathcal{F}_m^{p,q} = R^p \pi_* (\Omega_{P \times B/B}^q(m) \otimes I_{\mathfrak{X}})$ and of the Betti syzygy space $T_m^{p,q}$ are named similarly.

Remark 1.2 In Definition 1.1, only in the case of $p = 1$, the spaces $T_m^{p,q}$ are relating definitely to the “proper” Betti syzygy spaces (= “minimal generators” of the syzygies) by $T_m^{1,q}(b) \cong (\mathbb{Z}_{X(b)}^{(q)} / S_+ \cdot \mathbb{Z}_{X(b)}^{(q)})_{(m)}$ and $\beta_{q,m}(X(b)) = \dim T_m^{1,q}(b)$ ($q \geq 1$), where S_+ and $\mathbb{Z}_{X(b)}^{(q)}$ denote the irrelevant maximal ideal of $S = \mathbb{C}[Z_0, \dots, Z_N]$ and the q -th syzygy module of the homogeneous coordinate ring $R_{X(b)}$ of the closed fiber $X(b)$, namely $\mathbb{Z}_{X(b)}^{(q)} = \text{Im}[F_{X(b),q-1} \leftarrow F_{X(b),q}]$ for a minimal graded S -free resolution $F_{X(b),\bullet}$ of $R_{X(b)}$, respectively. In the cases of $p \neq 1$, depending on the arithmetic depth of $R_{X(b)}$, the spaces $T_m^{p,q}(b)$ are considered as objects relating vaguely to the $(q-p+1, m)$ -Betti syzygy spaces $(\mathbb{Z}_{X(b)}^{(q-p+1)} / S_+ \cdot \mathbb{Z}_{X(b)}^{(q-p+1)})_{(m)}$ through the Lefschetz maps $L = \cup c_1(O_P(1))$. For example, $L : T_m^{1,q}(b) \hookrightarrow T_m^{2,q+1}(b)$ in general. But if the closed fiber $X(b)$ satisfies the arithmetic D_3 -condition, e.g. arithmetically Cohen-Macaulay of $\dim X(b) \geq 2$, then $T_m^{1,q}(b) \cong T_m^{2,q+1}(b)$ and $L : T_m^{2,q+1}(b) \hookrightarrow T_m^{3,q+2}(b)$.

Remark 1.3 Take a closed point b_0 in an algebraic scheme B over \mathbb{C} and put \overline{B}_ν to be the ν -th infinitesimal neighborhood of b_0 in B . When we start from a given projective and flat family $g : \mathfrak{Y} \rightarrow \overline{B}_\nu$ over the fat point \overline{B}_ν , by putting $f = g$, $\mathfrak{X} = \mathfrak{Y}$, $B = \overline{B}_\nu$, b_0 to be the unique closed point of \overline{B}_ν in Circumstances 1.1 (AD2) of [9], since $(\overline{B}_\nu)_\nu = \overline{B}_\nu$, the induced family $\overline{f}_\nu : \overline{\mathfrak{X}}_\nu \rightarrow \overline{B}_\nu$ from this $f : \mathfrak{X} \rightarrow B$ coincides with the original family $g : \mathfrak{Y} \rightarrow \overline{B}_\nu$, to which we can apply Definition 1.1 above and have $\mathcal{F}_m^{p,q}(\mathfrak{Y}) \cong \overline{\mathcal{F}}_{m,\nu}^{p,q}(b_0)$. Thus the over line is used only to emphasize the objects being infinitesimal ones.

In [11], we introduced the concept “ q_0 -Bett constancy” for the inductive construction of the families with the (full) Betti constancy. Here we introduce a refinement of this concept for studying generally degenerations of syzygies.

Definition 1.4 ((q_0, m_0) -Betti constancy) Take an AD2-family $f : \mathfrak{X} \rightarrow B$ in $\mathbb{P}^N(\mathbb{C}) = P$, and integers q_0, m_0 with $0 \leq q_0 \leq N$ and fix them. Then we say that the family $f : \mathfrak{X} \rightarrow B$ is (q_0, m_0) -Betti constant if the coherent sheaves $\mathcal{F}_{m_0}^{1,q}(\mathfrak{X}) = R^1 \pi_* (\Omega_{P \times B/B}^q(m_0) \otimes I_{\mathfrak{X}})$ are O_B -locally free sheaves for all the integers q with $0 \leq q \leq q_0$. For the ν -th infinitesimal neighborhood \overline{B}_ν of a closed point $b_0 \in B$, this definition can be applied also to an AD2 family $g : \mathfrak{Y} \rightarrow \overline{B}_\nu$ including the case $\mathfrak{Y} = \overline{\mathfrak{X}}_\nu = \mathfrak{X} \times_B \overline{B}_\nu$ and $g = \overline{f}_\nu$ by using the modules $\{\overline{\mathcal{F}}_{m_0,\nu}^{1,q} | 0 \leq q \leq q_0\}$, since $\overline{\mathcal{F}}_{m_0,\nu}^{p,q}(b_0) = \mathcal{F}_m^{p,q}(\overline{\mathfrak{X}}_\nu)$. Similarly, the localized concept “ (q_0, m_0) -Betti constant around the point b_0 ” is defined by using the freeness of the stalks $\{(\mathcal{F}_{m_0}^{1,q})_{b_0} | 0 \leq q \leq q_0\}$ at the point b_0 , namely, the condition $(L.F.)_{m_0}^{1,q}(b_0)$ in [9] holds for $0 \leq q \leq q_0$.

Remark 1.5 For any integer $m_0 \in \mathbb{Z}$ including negative one, any AD2-family $f : \mathfrak{X} \rightarrow B$ in $\mathbb{P}^N(\mathbb{C})$ satisfies $(0, m_0)$ -Betti constancy (cf. (1.6.1) of Theorem 1.6 below).

Using Remark 1.5 above, we can amend Theorem 1.7 of [9] as follows since the inductive proof on q_0 for Theorem 1.7 in [9] does not need to run the polynomial degree m except the case $q = 0$.

Theorem 1.6 (cf. [11]) Let $f : \mathfrak{X} \rightarrow B$ be an AD2-family in $\mathbb{P}^N(\mathbb{C}) = P$. Then we see the following two properties.

(1.6.1) For any integer $m \in \mathbb{Z}$, the coherent sheaf $\mathcal{T}_m^{0,0} = \pi_*(I_{\mathfrak{X}}(m))$ is O_B -locally free.

(1.6.2) For any closed point $b \in B^{(cl.)}$ and any integer $m \in \mathbb{Z}$, the cohomological base change map $\varphi_m^{0,0}(b) : \mathcal{T}_m^{0,0} \otimes k(b) = \pi_*(I_{\mathfrak{X}}(m)) \otimes k(b) \rightarrow T_m^{0,0}(b) = H^0(I_{X(b)}(m))$ at the point b is an isomorphism.

Now we fix integers $q_0 \geq 0$ and m_0 . Assume moreover that the family $f : \mathfrak{X} \rightarrow B$ is (q_0, m_0) -Betti constant. Then we have the following two properties.

(1.6.3) The coherent sheaves $\mathcal{T}_{m_0}^{0,q} = \pi_*(\Omega_{P \times B/B}^q(m_0) \otimes I_{\mathfrak{X}})$ are O_B -locally free for all the integers $q \in \mathbb{Z}$ with $q_0 \geq q \geq 0$.

(1.6.4) If $p = 0$ or $p = 1$, then, for any closed point $b \in B^{(cl.)}$ and for any integer $q \in \mathbb{Z}$ with $q_0 \geq q \geq 0$, the cohomological base change map $\varphi_{m_0}^{p,q}(b) : \mathcal{T}_{m_0}^{p,q} \otimes k(b) = R^p \pi_*(\Omega_{P \times B/B}^q(m_0) \otimes I_{\mathfrak{X}}) \otimes k(b) \rightarrow T_{m_0}^{p,q}(b) = H^p(\Omega_{P_k(b)}^q(m_0) \otimes I_{X(b)})$ at the point b is an isomorphism. Moreover, if $p = 1$, $q = q_0 + 1$ and $b \in B^{(cl.)}$, then the map $\varphi_{m_0}^{p,q}(b)$ is still isomorphic.

In the introduction of [9], we described what is the ‘‘degeneration of syzygies’’ by using the Hilbert schemes. However, from the technical view point, that definition on the degeneration of syzygies is not so convenient to handle. Here we give a new definition on the ‘‘degeneration of syzygies’’ as a technical refinement of the previous one, which is using our previous definition on ‘‘ (q_0, m_0) -Betti constancy’’.

Definition 1.7 (degeneration of syzygies) Fix integers q_1, m_0 with $q_1 \geq 1$. We say that at a closed point $b_0 \in B$, an AD2-family $f : \mathfrak{X} \rightarrow B$ in $\mathbb{P}^N(\mathbb{C})$ has the degeneration of syzygies in minimal syzygy level q_1 and in S -degree m_0 (or simply, the degeneration of (q_1, m_0) -syzygies), if around the point b_0 , the family is $(q_1 - 1, m_0)$ -Betti constant but is not (q_1, m_0) -Betti constant. It is equivalent to the condition that $(L.F.)_{m_0}^{1,q}(b_0)$ holds for $0 \leq q \leq q_1 - 1$ and does not hold for $q = q_1$ (cf. Notation and Conventions 1.2 in [9]).

Remark 1.8 We consider that a zero module is a free module of rank zero as usual. Thus, when we say that a non-free module is given, then it implies that the module is a non-zero module. In particular, the stalk $(\mathcal{T}_{m_0}^{1,q_1})_{b_0}$ for the case $q = q_1$ in Definition 1.7 is not a zero module.

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Setup of Main Theorem

Now we describe the setup of Main Theorem 1.16 below. Fix the infinitesimal level ν to be 1 and abbreviate the index ν in the sequel. To clarify the assumption of Theorem 1.16, we make a mark \bullet at the head of the sentence describing the assumption whenever we add a new one.

• Let us take a projective scheme X of dimension $n \geq 1$ with $H^0(O_X) \cong \mathbb{C}$ and one of its arithmetic D_2 -embeddings $j : X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = Proj(S) = P$, where $S = \mathbb{C}[Z_0, \dots, Z_N]$ is a polynomial ring of $N + 1$ variables.

We want to analyze the infinitesimal variation of syzygies of X depending on a tangent vector $v \in \Theta_{\mathcal{H}, [X]}$ at the closed point $[X]$ of the Hilbert scheme $\mathcal{H} = Hilb_P^{AX(m)}$. Since the 1-st infinitesimal embedded deformations of X in P :

$$\begin{array}{ccc} \bar{\mathcal{X}} = \bar{\mathcal{X}}(\sigma) & \xrightarrow{\bar{j}} & P \times \bar{B} \\ & \searrow \bar{f} & \downarrow \bar{\pi} \\ & & \bar{B}. \end{array} \quad (\#-1)$$

over $\bar{B} = Spec(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ are classified by global normal vector fields $\sigma \in H^0(N_X) \cong Hom_{O_P}(I_X, O_X)$, by the universality of the Hilbert scheme \mathcal{H} , we see the well-known fact : $\Theta_{\mathcal{H}, [X]} \cong H^0(N_X)$.

• Let $\bar{f} : \bar{\mathcal{X}} \rightarrow \bar{B}$ be a 1-st infinitesimal embedded deformation of X in $\mathbb{P}^N(\mathbb{C}) = P$ which corresponds to a global normal vector field $\sigma \in H^0(N_X)$.

Thus we want to study the effect of the global normal vector field $\sigma \in H^0(N_X)$ to the structures of infinitesimal Betti syzygy modules $\{\bar{\mathcal{F}}_m^{p,q}\}$ and related maps (cf. (#-3) below). Let us recall the first row in the diagram (#-3) of [7]:

$$0 \longrightarrow I_X \xrightarrow{\times \varepsilon} I_{\bar{\mathcal{X}}} \longrightarrow I_X \longrightarrow 0. \quad (\#-2)$$

Tensoring $\Omega_{P \times \bar{B}/\bar{B}}^q(m)$ to this sequence (#-2) and taking higher direct images by $\bar{\pi}$, we have a long exact sequence of finite $O_{\bar{B}}$ -modules :

$$\longrightarrow T_m^{p-1,q} \xrightarrow{(ob_\sigma)_m^{p-1,q}} T_m^{p,q} \xrightarrow{\mu_m^{p,q}} \bar{\mathcal{F}}_m^{p,q} \xrightarrow{\lambda_m^{p,q}} T_m^{p,q} \xrightarrow{(ob_\sigma)_m^{p,q}} T_m^{p+1,q} \longrightarrow, \quad (\#-3)$$

where $T_m^{p,q} = H^p(\Omega_P^q(m) \otimes I_X)$, $\bar{\mathcal{F}}_m^{p,q} = R^p \bar{\pi}_*(\Omega_{P \times \bar{B}/\bar{B}}^q(m) \otimes I_{\bar{\mathcal{X}}})$, and the obstruction map $(ob_\sigma)_m^{p,q}$ is the same as the map $\delta_{IDF}^{(p)}$ in [7] for the case $F = \Omega_P^q(m)$. The map $\lambda_m^{p,q}$ is a composition map of the canonical map $\bar{\mathcal{F}}_m^{p,q} \rightarrow \bar{\mathcal{F}}_m^{p,q} \otimes k(b_0)$ and the cohomological base change map $\bar{\varphi}_m^{p,q}(b_0) : \bar{\mathcal{F}}_m^{p,q} \otimes k(b_0) \rightarrow T_m^{p,q}$ at the unique closed point $b_0 \in \bar{B}$, where $k(b_0) \cong \mathbb{C}$ denotes the residue field of the local ring $O_{\bar{B}}$.

Definition 1.9 (infinitesimally unstable Betti syzygy class) For integers p' , q' and m' , we consider the map $\lambda_{m'}^{p',q'} : \overline{\mathcal{F}}_{m'}^{p',q'} \rightarrow T_{m'}^{p',q'}$ in the sequence (#-3) and a non-zero class $\alpha \in T_{m'}^{p',q'}$. We say that the Betti syzygy class α is infinitesimally unstable in the direction of σ if the class α is obstructed (i.e. $(ob_\sigma)_{m'}^{p',q'}(\alpha) \neq 0$ in $T_{m'}^{p'+1,q'}$), or there exists an element $\tilde{\alpha} \in \overline{\mathcal{F}}_{m'}^{p',q'}$ such that $\varepsilon \cdot \tilde{\alpha} = 0$ and the class $\tilde{\alpha}$ is a $\lambda_{m'}^{p',q'}$ -lift of the class α (i.e. $\lambda_{m'}^{p',q'}(\tilde{\alpha}) = \alpha$). On the other hand, if the Betti syzygy class α is not infinitesimally unstable in the direction of σ , we say that the Betti syzygy class α is infinitesimally stable in the direction of σ . If the class α is infinitesimally stable in the direction of σ for any $\sigma \in H^0(N_X)$, we say that the Betti syzygy class α is infinitesimally stable in all directions.

Remark 1.10 In the Definition 1.9, if $(ob_\sigma)_{m'}^{p',q'}(\alpha) = 0$, to determine the infinitesimal unstability or stability of the class α in the direction of σ , it is enough to test $\varepsilon \cdot \tilde{\alpha} = 0$ or $\neq 0$ only for one $\lambda_{m'}^{p',q'}$ -lift $\tilde{\alpha}$ of the class α . To see this, let us take any other $\lambda_{m'}^{p',q'}$ -lift $\tilde{\beta}$ of the class α . Then recalling the sequence (#-3), we have a class $\delta \in T_{m'}^{p',q'}$ and $\tilde{\alpha} - \tilde{\beta} = \mu_{m'}^{p',q'}(\delta)$. Since the Betti syzygy space $T_{m'}^{p',q'}$ is a $k(b_0)$ -vector space and is annihilated by $\varepsilon \in O_{\overline{B}}$, we have

$$\varepsilon \cdot \tilde{\alpha} = \varepsilon \cdot (\tilde{\beta} + \mu_{m'}^{p',q'}(\delta)) = \varepsilon \cdot \tilde{\beta} + \mu_{m'}^{p',q'}(\varepsilon \cdot \delta) = \varepsilon \cdot \tilde{\beta}.$$

• Suppose that the family $\overline{f} : \overline{\mathcal{X}} \rightarrow \overline{B}$ has the degeneration of syzygies in minimal syzygy level q_1 and in S-degree m_0 at the point b_0 .

Applying (1.6.4) of Theorem 1.6, for any integer q with $q_1 \geq q \geq 0$, we obtain a shortened exact sequence from the sequence (#-3) :

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{m_0}^{0,q} & \xrightarrow[\times\varepsilon]{\mu_{m_0}^{0,q}} & \overline{\mathcal{F}}_{m_0}^{0,q} & \xrightarrow{\lambda_{m_0}^{0,q}} & T_{m_0}^{0,q} \\ & & & & & & \\ & & \xrightarrow{(ob_\sigma)_{m_0}^{0,q}} & T_{m_0}^{1,q} & \xrightarrow[\mu_{m_0}^{1,q}]{\times\varepsilon} & \overline{\mathcal{F}}_{m_0}^{1,q} & \xrightarrow{\lambda_{m_0}^{1,q}} & T_{m_0}^{1,q} & \xrightarrow{(ob_\sigma)_{m_0}^{1,q}} & 0. \end{array} \quad (\#-4)$$

Moreover, for an integer q with $q_1 - 1 \geq q \geq 0$, Theorem 1.6 implies also the surjectivity of the map $\lambda_{m_0}^{0,q}$, which shows that the map $(ob_\sigma)_{m_0}^{0,q}$ is a zero map and the sequence (#-4) is decomposed into two short exact sequences. On the other hand, if we assume that the map $(ob_\sigma)_{m_0}^{0,q}$ is zero, then it brings that the $O_{\overline{B}}$ -freeness of the two modules $\overline{\mathcal{F}}_{m_0}^{0,q}$ and $\overline{\mathcal{F}}_{m_0}^{1,q}$. Thus, in case of $q = q_1$, the map $(ob_\sigma)_{m_0}^{0,q}$ is not a zero map, which implies that $(C.B.C)_{m_0}^{0,q_1}$ does not hold, or equivalently $(C.B.C)_{m_0}^{1,q_1+1}$ does not hold (cf. Proposition 1.8 of [9]). Thus, what we want to see is the relation of the two spaces $T_{m_0}^{1,q_1}$ and $T_{m_0}^{1,q_1+1}$, e.g. comparing the graded Betti numbers $\beta_{q_1, m_0} = \dim T_{m_0}^{1,q_1}$ and $\beta_{q_1+1, m_0} = \dim T_{m_0}^{1,q_1+1}$.

Let us consider the sequence (#-4) more precisely in the case $q = q_1$. Since the ring $O_{\overline{B}}$ is a very special Artinian ring (i.e. a homomorphic image of a D.V.R.), we use here the technical term ‘‘rank’’ in a slightly different meaning from the usual one (cf. Definition 1.4.2. in [1]).

Lemma 1.11 *Under the circumstances, putting $r = \text{rank}_{O_{\overline{B}}} \overline{\mathcal{F}}_{m_0}^{1,q_1}$, which means the maximal rank of the $O_{\overline{B}}$ -free direct summands of the module $\overline{\mathcal{F}}_{m_0}^{1,q_1}$, and $s = \dim \text{Im}((ob_{\sigma})_{m_0}^{0,q_1})$, we have*

$$\beta_{q_1, m_0} = r + s, \quad s > 0.$$

Proof. Since $O_{\overline{B}} \cong \mathbb{C}[[\varepsilon]]/(\varepsilon^2)$, we can consider the module $\overline{\mathcal{F}}_{m_0}^{1,q_1}$ as a finite $\mathbb{C}[[\varepsilon]]$ -module annihilated by ε^2 via the $O_{\overline{B}}$ -module structure, where $\mathbb{C}[[\varepsilon]]$ denotes the formal power series ring of 1-variable ε . Using the fact that the ring $\mathbb{C}[[\varepsilon]]$ is a P.I.D. (or a D.V.R.), we see

$$\overline{\mathcal{F}}_{m_0}^{1,q_1} \cong \left(\bigoplus_{i=1}^r O_{\overline{B}} e_i \right) \oplus \left(\bigoplus_{i=r+1}^{r+s} k(b_0) e_i \right). \quad (\#-5)$$

These numbers r and s are recovered from the module $M = \overline{\mathcal{F}}_{m_0}^{1,q_1}$ through $r = \dim_{k(b_0)} \text{Im}[M \xrightarrow{\times \varepsilon} M]$ and $s = \dim_{k(b_0)} \text{Tor}_1^{O_{\overline{B}}}(M, k(b_0))$. In our proof, we set the number “s” to be as in (#-5) and show $s = \dim \text{Im}((ob_{\sigma})_{m_0}^{0,q_1})$ in the sequel. Since Theorem 1.6 shows $(C.B.C)_{m_0}^{1,q_1}(b_0)$, the map $\lambda_{m_0}^{1,q_1}$ induces the isomorphic cohomological base change map $\overline{\varphi}_{m_0}^{1,q_1} : \overline{\mathcal{F}}_{m_0}^{1,q_1} \otimes k(b_0) \rightarrow T_{m_0}^{1,q_1}$, which implies that the set $\{\lambda_{m_0}^{1,q_1}(e_i)\}_{i=1}^{r+s}$ forms a $k(b_0)$ -basis of the $k(b_0)$ -vector space $T_{m_0}^{1,q_1}$ and $\beta_{q_1, m_0} = r + s$. Thus, by a suitable base change of the $k(b_0)$ -vector space $T_{m_0}^{1,q_1}$, we can identify the map $\lambda_{m_0}^{1,q_1}$ with the canonical map arising from the natural maps $O_{\overline{B}} \rightarrow k(b_0) = O_{\overline{B}}/(\varepsilon)$ and $Id : k(b_0) \rightarrow k(b_0)$. Then $\text{Ker}(\lambda_{m_0}^{1,q_1}) = \bigoplus_{i=1}^r \mathbb{C}\varepsilon e_i = \text{Im}(\mu_{m_0}^{1,q_1})$. Now we see that $\dim \text{Im}((ob_{\sigma})_{m_0}^{0,q_1}) = \dim \text{Ker}(\mu_{m_0}^{1,q_1}) = \dim T_{m_0}^{1,q_1} - \dim \text{Im}(\mu_{m_0}^{1,q_1}) = \beta_{q_1, m_0} - r = s$. By our assumption of the degeneration of (q_1, m_0) -syzygies, the module $\overline{\mathcal{F}}_{m_0}^{1,q_1}$ itself is not $O_{\overline{B}}$ -free, we see that $s > 0$. ■

Let us introduce two additional concepts with certain geometric meanings, which may be suggested by their naming.

Definition 1.12 *Under the circumstances above, we give two definitions.*

(1.12.1) *We say that the normal vector field $\sigma \in H^0(N_X)$ is in the transversal direction only to the (q_1, m_0) -Betti constancy if $r = 0$, namely the module $\overline{\mathcal{F}}_{m_0}^{1,q_1}$ has a $k(b_0)$ -module structure via the original $O_{\overline{B}}$ -module structure, or equivalently, $\text{Ann}_{O_{\overline{B}}}(\overline{\mathcal{F}}_{m_0}^{1,q_1}) = (\varepsilon)$.*

(1.12.2) *For integers p' , q' and m' , if the Betti syzygy space $T_{m'}^{p',q'}$ is not zero and the map $\lambda_{m'}^{p',q'}$ is zero, then we say that the Betti syzygy space $T_{m'}^{p',q'}$ is totally obstructed.*

Lemma 1.13 *Under the circumstances above, the normal vector field σ is in the transversal direction only to the (q_1, m_0) -Betti constancy if and only if the obstruction map $(ob_{\sigma})_{m_0}^{0,q_1}$ is surjective.*

Proof. By Lemma 1.11, it is obvious because each of the two conditions in this Lemma is equivalent to $\mu_{m_0}^{1,q_1} = 0$. ■

Lemma 1.14 *Under the circumstances above, the following four conditions are equivalent.*

- (1.14.1) *The normal vector field σ is in the transversal direction only to the (q_1, m_0) -Betti constancy and the space $T_{m_0}^{0, q_1}$ is totally obstructed. (Hence $(C.B.C)_{m_0}^{0, q_1}$ does not hold in this case).*
- (1.14.2) *The maps $\lambda_{m_0}^{1, q_1}$ and $\mu_{m_0}^{0, q_1}$ are isomorphic.*
- (1.14.3) *The maps $\mu_{m_0}^{1, q_1}$ and $\lambda_{m_0}^{0, q_1}$ are zero.*
- (1.14.4) *The map $(ob_\sigma)_{m_0}^{0, q_1}$ is isomorphic.*

Proof. By the sequence (#-4) with putting $q = q_1$ and Theorem 12.11 of Cap.III in [3], the implications (1.14.1) \Rightarrow (1.14.2) \Rightarrow (1.14.3) \Rightarrow (1.14.4) are obvious. Now we assume (1.14.4) and show (1.14.1). Since the finite $O_{\bar{B}}$ -module $\overline{\mathcal{F}}_{m_0}^{1, q_1}$ is not zero (cf. Remark 1.8), Nakayama's lemma implies $T_{m_0}^{1, q_1} \cong \overline{\mathcal{F}}_{m_0}^{1, q_1} \otimes k(b_0) \neq 0$. By the isomorphism $(ob_\sigma)_{m_0}^{0, q_1}, T_{m_0}^{0, q_1} \cong T_{m_0}^{1, q_1} \neq 0$. Both of the two conditions (1.14.3) and (1.14.2) are also obtained easily. Thus, the map $\lambda_{m_0}^{0, q_1}$ is zero and the map $\lambda_{m_0}^{1, q_1}$ is isomorphic, and we have (1.14.1). ■

Let us summarize our assumptions in our setup above.

Assumption 1.15 *For Main Theorem 1.16, we need the following three conditions in our setup.*

- (1.15.1) *An embedding $j : X \hookrightarrow \mathbb{P}^N(\mathbb{C}) = P$ satisfies the arithmetic D_2 -condition.*
- (1.15.2) *$\bar{f} : \bar{\mathcal{X}} \rightarrow \bar{B}$ is a 1-st infinitesimal embedded deformation of X in P which corresponds to a global normal vector field $\sigma \in H^0(N_X)$.*
- (1.15.3) *The family $\bar{f} : \bar{\mathcal{X}} \rightarrow \bar{B}$ has the degeneration of (q_1, m_0) -syzygies.*

Main Theorem 1.16 (adjacent concurrence) *Suppose the three conditions in Assumption 1.15 above. Then, putting $s = \dim \text{Im}((ob_\sigma)_{m_0}^{0, q_1})$, we have the following three claims.*

- (1.16.1) *On the graded Betti numbers of X , we have inequalities $\beta_{q_1, m_0} \geq s > 0$ and $\beta_{q_1+1, m_0} \geq s > 0$. Moreover, each of the Betti syzygy spaces $T_{m_0}^{1, q_1}$ and $T_{m_0}^{1, q_1+1}$ has an infinitesimally unstable Betti syzygy class in the direction of σ .*
- (1.16.2) *If the normal vector field σ is in the transversal direction only to the (q_1, m_0) -Betti constancy, then we have the inequalities : $0 < \beta_{q_1, m_0} \leq \beta_{q_1+1, m_0}$.*
- (1.16.3) *Adding to the assumption of (1.16.2), if the space $T_{m_0}^{0, q_1}$ is totally obstructed, then we have $T_{m_0}^{1, q_1+1} \cong T_{m_0}^{0, q_1} \cong T_{m_0}^{1, q_1} \neq 0$, $0 < \beta_{q_1, m_0} = \beta_{q_1+1, m_0}$, and the map $(ob_\sigma)_{m_0}^{1, q_1+1}$ is injective, or equivalently the map $\lambda_{m_0}^{1, q_1+1} : \overline{\mathcal{F}}_{m_0}^{1, q_1+1} \rightarrow T_{m_0}^{1, q_1+1}$ is a zero map. In other words, the Betti syzygy space $T_{m_0}^{1, q_1+1}$ is totally obstructed.*

Proof. Let us recall a short exact sequence arising from the $(q_1 + 1)$ -exterior product of the mid-term in the Euler sequence for the sheaf of the relative differential 1-forms $\Omega_{P \times \overline{B}/\overline{B}}^1$ on $P \times \overline{B}$ over \overline{B} and from the induced filtration :

$$0 \longrightarrow \Omega_{P \times \overline{B}/\overline{B}}^{q_1+1}(m_0) \longrightarrow \bigoplus_{\binom{N+1}{q_1+1}} O_{P \times \overline{B}/\overline{B}}(m_0 - q_1 - 1) \longrightarrow \Omega_{P \times \overline{B}/\overline{B}}^{q_1}(m_0) \longrightarrow 0 \quad (\#-6)$$

with m_0 -tuple twisting by the relative ample line bundle $O_{P \times \overline{B}/\overline{B}}(1)$. Since every term in the sequence (#-6) is an $O_{P \times \overline{B}}$ -locally free sheaf, by tensoring two short exact sequences (#-6) and (#-2), we obtain a 3×3 exact commutative diagram. Taking higher direct images by the morphism $\overline{\pi}$ of each entry in this 3×3 exact commutative diagram with putting $k_0 = m_0 - q_1 - 1$, we get an exact commutative diagram:

$$\begin{array}{ccccc} & & & \oplus H^1(I_X(k_0)) = 0 & \\ & & & \downarrow & \\ \overline{\mathcal{F}}_{m_0}^{0,q_1} & \xrightarrow{\lambda_{m_0}^{0,q_1}} & T_{m_0}^{0,q_1} & \xrightarrow{(ob_\sigma)_{m_0}^{0,q_1}} & T_{m_0}^{1,q_1} \\ \gamma_0 \downarrow & & \gamma_1 \downarrow & & \downarrow \gamma_2 \\ \overline{\mathcal{F}}_{m_0}^{1,q_1+1} & \xrightarrow{\lambda_{m_0}^{1,q_1+1}} & T_{m_0}^{1,q_1+1} & \xrightarrow{(ob_\sigma)_{m_0}^{1,q_1+1}} & T_{m_0}^{2,q_1+1} \\ & & \downarrow & & \\ & & \oplus H^1(I_X(k_0)) = 0, & & \end{array} \quad (\#-7)$$

which implies an equality of the maps:

$$\gamma_2 \circ (ob_\sigma)_{m_0}^{0,q_1} = (ob_\sigma)_{m_0}^{1,q_1+1} \circ \gamma_1, \quad (\#-8)$$

and the injectivity of the map γ_2 and the surjectivity of the map γ_1 .

From Lemma 1.11, the inequality $\beta_{q_1, m_0} \geq s > 0$ is obvious. The other inequality in the claim (1.16.1) is obtained by

$$\begin{aligned} s &= \dim \operatorname{Im}(ob_\sigma)_{m_0}^{0,q_1} = \dim \operatorname{Im}(\gamma_2 \circ (ob_\sigma)_{m_0}^{0,q_1}) \\ &= \dim \operatorname{Im}((ob_\sigma)_{m_0}^{1,q_1+1} \circ \gamma_1) = \dim \operatorname{Im}((ob_\sigma)_{m_0}^{1,q_1+1}) \\ &\leq \dim T_{m_0}^{1,q_1+1} = \beta_{q_1+1, m_0}. \end{aligned}$$

Let us show the existence of infinitesimally unstable Betti syzygy classes in the Betti syzygy spaces $T_{m_0}^{1,q_1}$ and $T_{m_0}^{1,q_1+1}$. On the case $T_{m_0}^{1,q_1}$, since the module $\overline{\mathcal{F}}_{m_0}^{1,q_1}$ has the structure described in (#-5), we can take a non-zero class $\tilde{\alpha} \in \overline{\mathcal{F}}_{m_0}^{1,q_1}$ which has 1 in all the $k(b_0)$ -components and 0 in all the $O_{\overline{B}}$ -components. Then, by putting $\alpha := \lambda_{m_0}^{1,q_1}(\tilde{\alpha})$, we see easily that $\varepsilon \cdot \tilde{\alpha} = 0$ and $\alpha \neq 0$, which means that the Betti syzygy space $T_{m_0}^{1,q_1}$ has an infinitesimally unstable Betti syzygy class α . On the case $T_{m_0}^{1,q_1+1}$, we note the diagram

(#-7). Since $0 < s = \dim \operatorname{Im}(ob_\sigma)_{m_0}^{0,q_1}$, we can choose a non-zero class $\delta \in \operatorname{Im}(ob_\sigma)_{m_0}^{0,q_1} \subseteq T_{m_0}^{1,q_1}$. By the injectivity of the map γ_2 , we have $\gamma_2(\delta) \neq 0$. Take an $(ob_\sigma)_{m_0}^{0,q_1}$ -lift $\tilde{\delta} \in T_{m_0}^{0,q_1}$, namely $(ob_\sigma)_{m_0}^{0,q_1}(\tilde{\delta}) = \delta$ and put $\beta = \gamma_1(\tilde{\delta}) \in T_{m_0}^{1,q_1+1}$. Then the $(q_1 + 1, m_0)$ -Betti syzygy class $\beta \in T_{m_0}^{1,q_1+1}$ is an infinitesimally unstable Betti syzygy class. In fact,

$$(ob_\sigma)_{m_0}^{1,q_1+1}(\beta) = (ob_\sigma)_{m_0}^{1,q_1+1} \circ \gamma_1(\tilde{\delta}) = \gamma_2 \circ (ob_\sigma)_{m_0}^{0,q_1}(\tilde{\delta}) = \gamma_2(\delta) \neq 0.$$

Thus we obtain the claim (1.16.1).

By Lemma 1.13, we see the surjectivity of the map $(ob_\sigma)_{m_0}^{0,q_1}$, which implies $\beta_{q_1, m_0} = s$. Then the claim (1.16.1) implies the inequalities $0 < \beta_{q_1, m_0} \leq \beta_{q_1+1, m_0}$

Now we assume moreover that the space $T_{m_0}^{0,q_1}$ is totally obstructed. Then, by Lemma 1.14, we see that the map $(ob_\sigma)_{m_0}^{0,q_1}$ is isomorphic. Using the injectivity of γ_2 , the map composed in the left hand side of the equality (#-8) is injective, and therefore the map composed in the right hand side and the map γ_1 are also injective. Hence the map γ_1 is isomorphic and we see $T_{m_0}^{1,q_1+1} \cong T_{m_0}^{0,q_1} \cong T_{m_0}^{1,q_1} \neq 0$. By using again the injectivity of the map $(ob_\sigma)_{m_0}^{1,q_1+1} \circ \gamma_1$, we see that the map $(ob_\sigma)_{m_0}^{1,q_1+1}$ is also injective. ■

§2 An Application.

As an application of the results in the previous section, on a local 1-parameter family of canonical curves of genus 5 with a trigonal curve as a central fiber which is handled continuously in [9] and [10], we will determine the module structure of $\overline{\mathcal{F}}_3^{1,2}$ and the cohomological base change map $\overline{\varphi}_3^{1,2}$ in this section. This is one of our remaining problems indicated by Remark 2.3 of [9]. As we saw in Table 2 of [9], the condition $(C.B.C)_3^{1,2}(b_0)$ does not hold if $\tau(\sigma) \neq 0$ (cf. on the map τ , see below). Hence, if $\tau(\sigma) \neq 0$, then there is no guarantee that $\overline{\mathcal{F}}_3^{1,2} \cong \mathcal{F}_3^{1,2} \otimes O_{\overline{B}}$ holds in general. In fact, $\overline{\mathcal{F}}_3^{1,2} \neq 0$ and $\overline{\mathcal{F}}_3^{1,2} \not\cong \mathcal{F}_3^{1,2} \otimes O_{\overline{B}} = 0$ as we will see in Theorem 2.1 (cf. also Table 2 in [9]).

To describe our results on this example, let us take the Hilbert scheme $\mathcal{H} = \operatorname{Hilb}_P^{A(m)}$ of $\mathbb{P}(\mathbb{C})^4 = P$ associated with a Hilbert polynomial $A(m) = 8m - 4$. All the canonical curves of genus 5 are included in the universal family $\mathcal{U} \rightarrow \mathcal{H}$ of \mathcal{H} . Take a trigonal canonical curve $X \subseteq P$ of genus 5 and a closed point $b_0 = [X] \in \operatorname{Hilb}_P^{A(m)}$ corresponding to the curve X . Then the scheme \mathcal{H} is smooth at the point b_0 . The closure of the set of all the closed points in \mathcal{H} corresponding to trigonal canonical curves with genus 5 in P form a divisor in \mathcal{H} , which is denoted by \mathcal{D} . The divisor \mathcal{D} has an analytic branch \mathcal{D}_0 which is smooth at the point b_0 . Then we take a locally closed affine smooth curve $B \subseteq \mathcal{H}$ which passes through the point b_0 and has the property $B \cap \mathcal{D} = \{b_0\}$ by removing finite other closed points of $B \cap \mathcal{D}$ from B if it is necessary. Then, the curve B induces a flat and projective family $f : \mathfrak{X} = \mathcal{U} \times_{\mathcal{H}} B \rightarrow B$. Since the smoothness of a fiber is an open condition, by applying Riemann-Roch theorem, we may assume that all the fibers are canonical curves of genus 5. Let us take the 1-st infinitesimal neighborhood \overline{B} of the point b_0 in B and its induced family $\overline{f} : \overline{\mathfrak{X}} \rightarrow \overline{B}$, which naturally corresponds to a section $\sigma \in H^0(N_X) \cong \Theta_{\mathcal{H}, b_0}$ up to \mathbb{C}^\times -multiplication, in other words, $\Theta_{\mathcal{H}, b_0} \supseteq \Theta_{\overline{B}, b_0} = \mathbb{C} \cdot \sigma$. The inclusions $b_0 \in \mathcal{D}_0 \subseteq \mathcal{H}$ induce the natural projection map $\tau : \Theta_{\mathcal{H}, b_0} \rightarrow N_{\mathcal{D}_0/\mathcal{H}, b_0}$ from the tangent space of \mathcal{H} at the point b_0 to the normal vector space of the smooth branch \mathcal{D}_0 in \mathcal{H} at the point b_0 . This map τ can be computed by the

composition of the natural maps $\tau : H^0(N_X) \rightarrow H^0(N_V \otimes O_X) \rightarrow H^1(N_V \otimes I_{X/V})$, where V denotes a unique non-singular cubic surface including the curve X (cf. [9]).

Let us recall Main Theorem 1.1 of [10] and its proof. If $\tau(\sigma) = 0$, then all the modules $\{\overline{\mathcal{T}}_3^{1,q} | q \geq 0\}$ are $O_{\overline{B}}$ -free. By Theorem 1.6, the cohomological base change maps $\{\overline{\varphi}_3^{p,q} | q \geq 0, p = 0, 1\}$ are isomorphic, which implies the equalities $(ob_\sigma)_3^{p,q} = 0$ ($q \geq 0, p = 0, 1$) and the surjectivity of the maps $\{\lambda_3^{p,q} | q \geq 0, p = 0, 1\}$. Thus, if $\tau(\sigma) = 0$, we were able to determine easily the module structure of $\overline{\mathcal{T}}_3^{1,2}$ and to see the cohomological base change map $\overline{\varphi}_3^{1,2}$ being isomorphic. Including the remaining case $\tau(\sigma) \neq 0$, we can summarize our results as follows.

Theorem 2.1 *Under the circumstances, the module structure of $\overline{\mathcal{T}}_3^{1,2}$ and the cohomological base change map $\overline{\varphi}_3^{1,2}$ are determined as follows.*

(2.1.1) *If $\tau(\sigma) = 0$, then $\overline{\mathcal{T}}_3^{1,2} \cong (O_{\overline{B}})^{\oplus 2}$, the cohomological base change map $\overline{\varphi}_3^{1,2}$ is an isomorphism.*

(2.1.2) *If $\tau(\sigma) \neq 0$, then $\overline{\mathcal{T}}_3^{1,2} \cong T_3^{1,2} \cong k(b_0)^{\oplus 2}$, where the map $\mu_3^{1,2} : T_3^{1,2} \rightarrow \overline{\mathcal{T}}_3^{1,2}$ is isomorphic. On the other hand, both of the map $\lambda_3^{1,2} : \overline{\mathcal{T}}_3^{1,2} \rightarrow T_3^{1,2}$ and the cohomological base change map $\overline{\varphi}_3^{1,2}$ are zero maps.*

Proof. For the claim (2.1.1), it is enough to recall the fact $\beta_{2,3}(X) = 2$ (cf. Table 2 in [9]) and the observation before Theorem 2.1.

Let us consider the claim (2.1.2) and assume $\tau(\sigma) \neq 0$. Recall the initial seven terms of the exact sequence (#-3) with putting $q = 2$ and $m = 3$,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_3^{0,2} & \xrightarrow[\times \varepsilon]{\mu_3^{0,2}} & \overline{\mathcal{T}}_3^{0,2} & \xrightarrow{\lambda_3^{0,2}} & T_3^{0,2} \\
 & & & & & & \\
 & & \longrightarrow & T_3^{1,2} & \xrightarrow[\mu_3^{1,2}]{\times \varepsilon} & \overline{\mathcal{T}}_3^{1,2} & \xrightarrow{\lambda_3^{1,2}} & T_3^{1,2} & \longrightarrow & T_3^{2,2} \\
 & & (ob_\sigma)_3^{0,2} & & & & & (ob_\sigma)_3^{1,2} & &
 \end{array} \tag{\#-9}$$

Since the natural inclusions $\{b_0\} \hookrightarrow \overline{B} \hookrightarrow B$ induce natural sheaf homomorphisms $I_{\mathfrak{X}} \rightarrow I_{\overline{\mathfrak{X}}} = I_{\mathfrak{X}} \otimes \pi^* O_{\overline{B}} \rightarrow I_X = I_{\mathfrak{X}} \otimes \pi^* k(b_0)$, where both of the two tensorings are taken over $\pi^* O_B$ with using the flatness of the sheaf $I_{\mathfrak{X}}$ over B , we have natural ‘‘sheaf’’ homomorphisms $\mathcal{T}_m^{p,q} \xrightarrow{\psi} \overline{\mathcal{T}}_m^{p,q} \xrightarrow{\lambda_m^{p,q}} T_m^{p,q}$, which imply a commutative diagram :

$$\begin{array}{ccccc}
 (\mathcal{T}_m^{p,q})_{b_0} & \xrightarrow{\text{can.}} & \mathcal{T}_m^{p,q} \otimes k(b_0) & & \\
 \downarrow \psi & & \downarrow \psi \otimes 1_{k(b_0)} & \searrow \varphi_m^{p,q} & \\
 \overline{\mathcal{T}}_m^{p,q} & \xrightarrow{\text{can.}} & \overline{\mathcal{T}}_m^{p,q} \otimes k(b_0) & \xrightarrow{\overline{\varphi}_m^{p,q}} & T_m^{p,q}.
 \end{array} \tag{\#-10}$$

Putting $p = 0, q = 2$, and $m = 3$ in the diagram (#-10) above, Table 2 in [9] tells us that $T_3^{1,2} \cong k(b_0)^{\oplus 2}$ and the condition $(C.B.C)_3^{0,2}(b_0)$ holds, which shows that the homomorphism $\varphi_3^{0,2}$ is isomorphic. Hence

we see the surjectivity of the maps $\overline{\varphi}_3^{0,2}$ and $\lambda_3^{0,2} = \overline{\varphi}_3^{0,2} \circ \text{can.}$, where “can.” denotes the canonical map. Then the exact sequence (#-9) gives

$$0 \longrightarrow T_3^{1,2} \xrightarrow[\mu_3^{1,2}]{\times \varepsilon} \overline{\mathcal{F}}_3^{1,2} \xrightarrow[\lambda_3^{1,2}]{\longrightarrow} T_3^{1,2} \xrightarrow[(ob_\sigma)_3^{1,2}]{\longrightarrow} T_3^{2,2}. \quad (\#-11)$$

Since $\lambda_3^{1,2} = \overline{\varphi}_3^{1,2} \circ \text{can.}$, by the exact sequence (#-11), it is enough to show the injectivity of the map $(ob_\sigma)_3^{1,2}$.

Now let us recall Theorem 1.1 of [10] and its proof, where we show the surjectivity of the map $ob_\sigma = (ob_\sigma)_3^{0,1} : T_3^{0,1} \rightarrow T_3^{1,1} \cong k(b_0)^{\oplus 2}$. We compute $\dim T_3^{0,1}$ by an exact sequence:

$$0 \longrightarrow T_3^{0,1} \longrightarrow \oplus^5 T_2^{0,0} \longrightarrow T_3^{0,0} \longrightarrow T_3^{1,1} \cong k(b_0)^{\oplus 2} \longrightarrow 0, \quad (\#-12)$$

which arises from the Euler sequence on P with tensoring $I_X(3)$. By the fact $O_X(1) \cong O_X(K_X)$ and arithmetically Cohen-Macaulay property of the homogeneous coordinate ring R_X , Riemann-Roch theorem shows $\dim T_3^{0,0} = 15$ and $\dim T_2^{0,0} = 3$. Thus we see that $\dim T_3^{0,1} = 2$ and the map $(ob_\sigma)_3^{0,1}$ is isomorphic. Then, Lemma 1.14 and the claim (1.16.3) of Main Theorem 1.16 in the previous section with setting $q_1 = 1$ and $m_0 = 3$ show the injectivity of the map $(ob_\sigma)_3^{1,2}$. ■

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