Families of canonical curves with genus 5 and the degenerations of the syzygies (I)

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Abstract

In the previous article [14], we obtained an inductive technique to show the cohomological base change property on syzygies only from the locally freeness of the first higher direct image sheaves for a flat family of arithmetic D_2-closed subschemes in a projective space. A slight improvement of this technique in [15] makes us possible to study the degeneration of syzygies, namely inconstancy of the graded Betti numbers. This technique can be applied to any arithmetic D_2 closed subscheme including a reducible singular variety X with $H^0(X, O_X) \cong \mathbb{C}$ and $\dim X > 0$, but we restrict ourselves to the case of a smooth projective variety having degenerate syzygies as the matter in hand for finding the essential difficulties in studying degenerate syzygies. Thus, as our first step, we give a preparatory study for applying our new technique to the degeneration of syzygies for a flat family of canonical curves with genus 5 in this article.

Keywords: canonical curve, genus 5, degeneration of syzygies

§0 Introduction.

Let us put $S = \mathbb{C}[Z_0, \ldots, Z_N]$ and consider a finitely generated graded S-module M. The graded Betti numbers $\{\beta_{q,m}(M)\}$ of the S-module M are defined by $\beta_{q,m}(M) = \dim_{\mathbb{C}} Tor_q^S(M, S/S_+)_{(m)}$ with using the irrelevant maximal ideal $S_+ = (Z_0, \ldots, Z_N)$ of S. For a closed subscheme $X \subseteq P = Proj(S)$, we set the graded Betti number $\beta_{q,m}(X)$ of X to be the one $\beta_{q,m}(R_X)$ of the homogeneous coordinate ring $R_X = S/\mathbb{I}_X$ of X, where $\mathbb{I}_X = \oplus_{m \geq 0} \Gamma(P, I_X(m))$.

For a flat family $f : \mathfrak{X} \to B$ of closed subschemes in the projective space P, the upper semi-continuity of their graded Betti numbers $\{\beta_{q,m}(f^{-1}(b))\}$ with respect to closed points $b \in B^{(cl.)}$ does not hold in general (cf. [15]). However, if we assume that those closed subschemes $X(b) := \{f^{-1}(b)\}_{b \in B^{(cl.)}}$ satisfy the arithmetic D_2-condition, the graded Betti number $\beta_{q,m}(X(b))$ is translated into the dimensions of the first cohomology group by $\beta_{q,m}(X(b)) = h^1(P, \Omega_P^q(m) \otimes I_{X(b)})$ and satisfies the upper semi-continuity.

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Now we take the Hilbert scheme $\mathcal{H} = Hilb_P^{A(m)}$ which parametrizes all the closed subschemes in P with a Hilbert polynomial $A(m)$ and its universal family $\Phi : \mathcal{U} \to \mathcal{H}$ over $\mathcal{H}$. If there exists a closed point b_0 such that the closed subscheme $X(b_0) := \Phi^{-1}(b_0) \subseteq P$ satisfies the arithmetic D_2-condition, then we can find a maximal open subscheme $\mathcal{H}_0 \subseteq \mathcal{H}$ including the point b_0 and for any closed point $b \in \mathcal{H}_0^{(cl.)}$, the fiber $X(b) := \Phi^{-1}(b)$ also satisfies the arithmetic D_2-condition. As we show in [15], for non-negative integers $\{c_{q,m}\}_{q \geq 1, m \geq 1}$, we have a locally closed subscheme $\mathcal{H}(\{c_{q,m}\}) \subseteq \mathcal{H}_0$ with a natural scheme structure such that the set $\mathcal{H}(\{c_{q,m}\})^{(cl.)}$ coincides with the set $\{b \in \mathcal{H}_0^{(cl.)} | \beta_{q,m}(X(b)) = c_{q,m} \ (q \geq 1, m \geq 1)\}$. Since the set $\{b \in \mathcal{H}_0^{(cl.)} | \beta_{q,m}(X(b)) \geq c_{q,m} \ (q \geq 1, m \geq 1)\}$ coincides with the set of closed points of a Zariski closed subset of $\mathcal{H}_0$ by the upper semi-continuity of the graded Betti numbers $\{\beta_{q,m}\}$, the scheme theoretic closure $\overline{\mathcal{H}(\{c_{q,m}\})}$ of the scheme $\mathcal{H}(\{c_{q,m}\})$ in the scheme $\mathcal{H}_0$ satisfies that for any closed point $b^* \in \overline{\mathcal{H}(\{c_{q,m}\})} - \mathcal{H}(\{c_{q,m}\})$, we have inequalities $\beta_{q,m}(X(b^*)) \geq c_{q,m}$ ($\forall q \geq 1$, $\forall m \geq 1$) and for some $q_0 \geq 1$ and $m_0 \geq 1$, we have a strict inequality : $\beta_{q_0,m_0}(X(b^*)) > c_{q_0,m_0}$. In this case, we say that the scheme $X(b^*)$ has the "degenerate" syzygies from those of $\mathcal{H}(\{c_{q,m}\})$.

Let us return back to the case of a flat family $f : \mathfrak{X} \to B$ of closed subschemes in the projective space P and assume that for any closed point $b \in B^{(cl.)}$, the closed subscheme $X(b)$ of P satisfies the arithmetic D_2-condition and has $A(m)$ as a Hilbert polynomial. Now we set π to be the second projection morphism $P \times B \to B$.

As the first step, let us consider the case that the higher direct image sheaf $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ is an O_B-locally free sheaf of rank $c_{q,m}$ for any integer $q \geq 1$ and $m \geq 1$. In this case, as we show in [15], there exists a unique morphism $\mu : B \to \mathcal{H}(\{c_{q,m}\}) \subseteq Hilb_P^{A(m)}$ with the scheme theoretic coincidence : $\mathfrak{X} = \mathcal{U} \times_{\mathcal{H}(\{c_{q,m}\})} B$ as closed subschemes of $P \times B$, and for any closed point $b \in B^{(cl.)}$, $\beta_{q,m}(X(b)) = c_{q,m}$ ($\forall q \geq 1$, $\forall m \geq 1$). In case of the base scheme B being reduced, the converse is also true, namely the Betti constancy $\beta_{q,m}(X(b)) = c_{q,m}$ ($\forall b \in B^{(cl.)}, \forall q \geq 1$, $\forall m \geq 1$) implies the locally freeness of the sheaf $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ ($\forall q \geq 1$, $\forall m \geq 1$) and the existence of the morphism μ by Grauert's theorem.

In the next step, instead of assuming the locally freeness of the sheaf $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ ($\forall q \geq 1$, $\forall m \geq 1$), we assume that the scheme B is an affine smooth curve and the sheaf $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ is locally free of rank $c_{q,m}$ outside of a closed point $b_0 \in B^{(cl.)}$, namely on the open set $B^\times = B - \{b_0\}$ ($\forall q \geq 1$, $\forall m \geq 1$), which is an easiest case of studying the degeneration of syzygies. Even in this case, by the universality of the Hilbert schemes, we have a unique morphism $\mu : B \to \overline{\mathcal{H}(\{c_{q,m}\})} \subseteq Hilb_P^{A(m)}$ and $\mathfrak{X} = \mathcal{U} \times_{\mathcal{H}(\{c_{q,m}\})} B$. Moreover, since the local ring O_{B,b_0} is a discrete valuation ring, the stalk $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})_{b_0}$ is isomorphic to an O_{B,b_0}-module $O_{B,b_0}^{\oplus r} \oplus (\oplus_{i=1}^s O_{B,b_0}/(t^{m_i}))$ where t denotes the regular parameter of the ring O_{B,b_0} and $m_i \in \mathbb{Z}_{\geq 1}$. However, we are confronted here with the difficulty of studying the degeneration of syzygies caused by the fact that the cohomological base change property of the sheaf $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ does not hold in general. Namely, it may happen that $h^1(\Omega^q_P(m) \otimes I_{X(b_0)}) \neq r + s$ and the module structure of the stalk $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})_{b_0}$ does not reflect the structure of the syzygies of the limit fiber $X(b_0)$ (cf. Remark 2.3).

In [15], we obtain a slight improvement of the technique of [14], which makes us possible to study the degeneration of syzygies in any degree but at the lowest level on "q" (cf. Theorem 1.7). This technique can be applied to any flat family of arithmetic D_2 closed subschemes including the case that the limit fiber $X(b_0)$ is a reducible singular variety having the properties : $H^0(X(b_0), O_{X(b_0)}) \cong \mathbb{C}$ and $\dim X(b_0) > 0$.

However, to exclude non-essential difficulties in our experimental study, we restrict ourselves to the case that the limit fiber $X(b_0)$ is a smooth projective variety with the degenerate syzygies, e.g. a trigonal canonical curve of genus 5 as the matter in hand. Fortunately, by virtue of the classical famous works [5], [6], [7], we have good knowledge on our case, namely the canonical curves of genus 5 including trigonal ones. In this article, we give a preparatory study on this subject including a principle for analyzing the degeneration of syzygies(cf. Theorem 1.4) and a conjecture (cf. Conjecture 2.4).

In this article, we refer fundamentally to [3] or to [2], and use the terminology and the results in [3] without mentioning except somethings important.

§1 Preliminaries.

We will work in the category of algebraic schemes over the complex number field $\mathbb{C}$. Namely, all the objects and the morphisms under consideration are algebraic schemes over $\mathbb{C}$ and (relative) morphisms of finite type with respect to $Spec(\mathbb{C})$.

In the sequel, we often consider the flat families of closed subschemes in a projective space in the following common circumstances and use the notation and conventions.

Circumstances 1.1 (AD2) *Let B a connected algebraic scheme over $\mathbb{C}$, $P = \mathbb{P}^N(\mathbb{C})$, and $f = \pi|_{\mathfrak{X}} : \mathfrak{X} \to B$ a projective and flat morphism as in the following diagram (#-1), where $\mathfrak{X}$ and "incl." denote a closed subscheme of $P \times B$ and an inclusion morphism, respectively.*
We assume that for any closed point $b \in B^{(cl.)}$, the fiber $X(b) = f^{-1}(b)$ is an arithmetic D_2-closed subscheme of P, namely $H^1(P, I_{X(b)}(m)) = 0$ $(\forall m \in \mathbb{Z})$.

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{incl.} & P \times B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi = pr_B} \\ & & B \end{array} \qquad (\#\text{-}1)$$

Taking a closed point $b_0 \in B$, the maximal ideal $\mathfrak{m}_{b_0}$ of the stalk O_{B,b_0}, and a non-negative integer ν, we set $\overline{B}_\nu = \overline{B}_\nu(b_0) = Spec(O_{B,b_0}/{\mathfrak{m}_{b_0}}^{\nu+1}) \subseteq B$, which is the ν-th infinitesimal neighborhood of the point b_0 in B. Then we make a fiber product $\overline{\mathfrak{X}}_\nu = \overline{\mathfrak{X}}_\nu(b_0) = \mathfrak{X} \times \overline{B}_\nu(b_0)$, which induces a diagram:

$$\begin{array}{ccc} \overline{\mathfrak{X}}_\nu & \xrightarrow{incl.} & P \times \overline{B}_\nu \\ & {\scriptstyle \overline{f}_\nu}\searrow & \downarrow{\scriptstyle \overline{\pi}_\nu} \\ & & \overline{B}_\nu . \end{array} \qquad (\#\text{-}2)$$

Notation and Conventions 1.2 *Under Circumstances 1.1, for non-negative integers* $p, q, m \in \mathbb{Z}_{\geq 0}$ *and a closed point* $b_0 \in B^{(cl.)}$*, we set*

$$\mathscr{T}_m^{p,q} = \mathscr{T}_m^{p,q}(\mathfrak{X}) := R^p\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}}),$$

$$\overline{\mathscr{T}}^{p,q}_{m,\nu}(b_0) := R^p(\overline{\pi_\nu})_*(\Omega^q_{P\times \overline{B}_\nu(b_0)/\overline{B}_\nu(b_0)}(m)\otimes I_{\overline{\mathfrak{X}}_\nu(b_0)}),$$

$$T_m^{p,q}(b_0) = T_m^{p,q}(X(b_0)) := H^p(\Omega^q_P(m)\otimes I_{X(b_0)}),$$

$$\varphi_m^{p,q}(b_0) : \mathscr{T}_m^{p,q}(\mathfrak{X})\otimes k(b_0) \longrightarrow T_m^{p,q}(X(b_0)) \quad (\textit{a natural homomorphism w.r.t. the base change}).$$

Fixing the integers p, q, m *and the closed point* $b_0 \in B^{(cl.)}$*, we introduce two abbreviations on the conditions of locally freeness and of the cohomological base change property.*

$(L.F.)_m^{p,q}(b_0)$ *: The sheaf* $\mathscr{T}_m^{p,q}(\mathfrak{X})$ *is of* O_B*-locally free in a suitable open neighborhood of the point* b_0*.*

$(C.B.C.)_m^{p,q}(b_0)$ *: The natural map* $\varphi_m^{p,q}(b_0) : \mathscr{T}_m^{p,q}(\mathfrak{X})\otimes k(b_0) \longrightarrow T_m^{p,q}(X(b_0))$ *is surjective, or equivalently bijective* (cf. Theorem 12.11 (a) in [3]).

Remark 1.3 *In case of* $\nu = 1$*, and if we have no risk of confusion, we often omit the description on the infinitesimal level* ν *and the point* b_0 *such as* $\overline{\mathscr{T}}^{p,q}_m$, $\overline{f}$, $\overline{\pi}$ *and so on.*

Next theorem might give a hint for analyzing an explicit module structure of $\mathscr{T}_m^{p,q}(\mathfrak{X})_{b_0}$, which can be considered as a partial refinement of "Theorem of Formal Functions" (cf. Theorem 11.1 Chap. III in [3]).

Theorem 1.4 (A principle for the structure analysis on $\mathscr{T}_m^{p,q}(\mathfrak{X})_{b_0}$) *Under Circumstances 1.1, let the base scheme* B *be a smooth connected curve. Set* $r := \dim_{k(\zeta)} \mathscr{T}_m^{p,q}(\mathfrak{X})_\zeta$*, where* ζ *denotes the generic point of the curve* B*. For some* $p, q, m, \nu \in \mathbb{Z}_{\geq 0}$*, and a closed point* $b_0 \in B^{(cl.)}$*, assume that the condition* $(C.B.C.)_m^{p,q}(b_0)$ *holds and the module* $\overline{\mathscr{T}}^{p,q}_{m,\nu}(b_0)$ *is isomorphic to* $[O_{B,b_0}/(t^{\nu+1})]^{\oplus r} \oplus (\oplus_{i=1}^s O_{B,b_0}/(t^{m_i}))$ *with* $0 \leq m_i \leq \nu$ *for* $i = 1, \ldots s$*, where " t " denotes a regular parameter of the discrete valuation ring* O_{B,b_0}*. Then, we have* $\mathscr{T}_m^{p,q}(\mathfrak{X})_{b_0} \cong O_{B,b_0}^{\oplus r} \oplus (\oplus_{i=1}^s O_{B,b_0}/(t^{m_i}))$*.*

Proof. Since the local ring O_{B,b_0} is a discrete valuation ring, the module $\mathscr{T}_m^{p,q}(\mathfrak{X})_{b_0}$ which is the stalk at the point b_0 must have the form $O_{B,b_0}^{\oplus r} \oplus (\oplus_{i=1}^{s'} O_{B,b_0}/(t^{k_i}))$. Then we may assume that we can choose non-negative integers $s(1)$ and $s(2)$ which satisfy $s' = s(1) + s(2)$, $0 \leq k_i \leq \nu$ ($0 \leq \forall i \leq s(1)$), and $k_j > \nu$ ($s(1) + 1 \leq \forall j \leq s'$).

Take a sufficiently small Zariski open neighborhood U of the point b_0 in B such that for any base extension morphism $u : D \to U \subseteq B$,

$$\begin{array}{ccc} P\times D & \xrightarrow{\ v\ } & P\times B \\ {\scriptstyle \pi_D}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow[u]{} & B \end{array} \qquad (\#\text{-}3)$$

we have a natural isomorphism : $u^*R^p\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}}) \overset{\sim}{\to} R^p(\pi_D)_*(\Omega^q_{P\times D/D}(m)\otimes I_{\mathfrak{Y}})$, where $\mathfrak{Y}=\mathfrak{X}\times_B D$ and $I_{\mathfrak{Y}}=v^*I_{\mathfrak{X}}\subseteq O_{P\times D}$ (cf. Lecture 7 in [4], or the proof in Chap.III §12 of [3]). Then we take $\overline{B}_\nu(b_0)$ as D, we see that $\mathscr{T}^{p,q}_m(\mathfrak{X})\otimes O_{\overline{B}_\nu(b_0)}\cong \overline{\mathscr{T}}^{p,q}_{m,\nu}(b_0)$. Our assumption implies that $\mathscr{T}^{p,q}_m(\mathfrak{X})\otimes O_{\overline{B}_\nu(b_0)}\cong (O_{B,b_0}/(t^{\nu+1}))^{\oplus r+s(2)}\oplus(\oplus_{i=1}^{s(1)}O_{B,b_0}/(t^{k_i}))\cong (O_{B,b_0}/(t^{\nu+1}))^{\oplus r}\oplus(\oplus_{i=1}^{s}O_{B,b_0}/(t^{m_i}))$. Thus we see that $s(2)=0$, $s(1)=s$ and the integers $k_1,k_2,\ldots,k_s$ coincide with the integers $m_1,m_2,\ldots,m_s$ up to permutations and with including multiplicities. ∎

Let us recall the definition of "q_0-Betti constancy" from [15], which is useful not only for constructing universal families of homological shells but also for studying degenerations of syzygies.

Definition 1.5 (q_0-Betti constancy cf.[15]) *Under Circumstances 1.1 (AD2), we take an integer q_0 with $0\le q_0\le N$ and fix it. Then we say that the family $f:\mathfrak{X}\to B$ is* q_0-Betti constant *if the coherent sheaves $\mathscr{T}^{1,q}_m(\mathfrak{X})=R^1\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ are O_B-locally free sheaves for all the integers q and m with $0\le q\le q_0$ and $m\ge 1$. In case of $q_0=N$, we simply say that the family $f:\mathfrak{X}\to B$ is* Betti constant.

Remark 1.6 *Under Circumstances 1.1 (AD2), every family $f:\mathfrak{X}\to B$ obviously satisfies* 0-Betti constancy *by our assumption.*

We refer to the following theorem in [15], which makes us possible to study the degeneration of (q_0+1)-syzygies of a family with q_0-Betti constancy.

Theorem 1.7 (cf. [15]) *Under Circumstances 1.1 (AD2), we suppose that the family $f:\mathfrak{X}\to B$ is q_0-Betti constant. Then we have the following four properties.*

(1.7.1) *The coherent sheaves $\pi_*(I_{\mathfrak{X}}(m))$ are O_B-locally free sheaves ($\forall m\in\mathbb{Z}$).*

(1.7.2) *For any closed point $b\in B$, the natural map $\varphi^{0,0}_m(b):\pi_*(I_{\mathfrak{X}}(m))\otimes k(b)\to H^0(I_{X(b)}(m))$ is an isomorphism ($\forall m\in\mathbb{Z}$).*

(1.7.3) *The coherent sheaves $\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ are O_B-locally free sheaves for all the integers $q,m\in\mathbb{Z}$ with $q_0\ge q\ge 0,\ m\ge 1$.*

(1.7.4) *If $p=0$ or $p=1$, for any closed point $b\in B^{(cl.)}$ and for all the integers $q,m\in\mathbb{Z}$ with $q_0\ge q\ge 0,\ m\ge 1$, the natural map $\varphi^{p,q}_m(b):R^p\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})\otimes k(b)\to H^p(\Omega^q_{P_{k(b)}}(m)\otimes I_{X(b)})$ is an isomorphism. Moreover, if $p=1,\ q=q_0+1,\ m\ge 1$ and $b\in B^{(cl.)}$, the map $\varphi^{p,q}_m(b)$ is still an isomorphism.*

Proposition 1.8 *Under Circumstances 1.1 (AD2), for* $m, q \in \mathbb{Z}_{\geq 1}$ *and a closed point* $b \in B^{(cl.)}$*, we have the following equivalence on the two conditions on the cohomological base change properties.*

$$(C.B.C.)_m^{0,q-1}(b) \quad \Leftrightarrow \quad (C.B.C.)_m^{1,q}(b).$$

Proof. Since the family $f : \mathfrak{X} \to B$ is 0-Betti constant, we have an exact commutative diagram :

$$\begin{array}{ccccccc}
\bigoplus \mathscr{T}_{m-q}^{0,0} \otimes k(b) & \longrightarrow & \mathscr{T}_m^{0,q-1} \otimes k(b) & \longrightarrow & \mathscr{T}_m^{1,q} \otimes k(b) & \longrightarrow & 0 \\
\cong \downarrow \varphi_{m-q}^{0,0}(b) & & \downarrow \varphi_m^{0,q-1}(b) & & \downarrow \varphi_m^{1,q}(b) & & \\
\bigoplus T_{m-q}^{0,0}(b) & \longrightarrow & T_m^{0,q-1}(b) & \longrightarrow & T_m^{1,q}(b) & \longrightarrow & 0
\end{array}$$

from Theorem 1.7 (1.7.2). It is easy to see the equivalence of the two conditions, namely the surjectivity of the map $\varphi_m^{1,q}(b)$ and that of the map $\varphi_m^{0,q-1}(b)$ by an easy diagram casing.

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§2 Some Results.

Let us recall what we want to study in the sequel. A canonical curve $X \subseteq \mathbb{P}^{g-1}(\mathbb{C}) = P$ of genus $g \geq 3$ is a non-hyper-elliptic smooth projective curve X with genus $g \geq 3$ which is embedded by the (complete) canonical linear system $|K_X|$ into the projective space $\mathbb{P}^{g-1}(\mathbb{C})$. We say that the curve X is trigonal if the curve X has a linear system g_3^1.

Now we restrict ourselves to the case $g = 5$, which gives an excellent experimental field of our study from the view point of syzygies (cf. [8] $\sim$ [12]). Since any trigonal canonical curve X of genus 5 arises from a plane quintic curve X_0 with one node p_0 (cf. [3] Chap. IV §4.5), by setting $p_0 = [1 : 0 : 0] \in Y = \mathbb{P}^2(\mathbb{C})$, we take a blow up $\beta : V \to Y$ of Y at the center p_0 and a line $L \cong \mathbb{P}^1(\mathbb{C})$ in Y outside of the point p_0. Then the projection $Y - \{p_0\} \to L$ extends uniquely to a morphism $\rho : V \to L$, which gives the surface V a $\mathbb{P}^1$-bundle structure $\mathbb{P}(O_L(-1) \oplus O_L) \to L$ arising from the vector bundle $F_0 = O_L(-1) \oplus O_L$ on L. Let ℓ be a line in Y passing through the point p_0, ξ the strict transform of ℓ, and ε the exceptional curve arising from the blow up β. Then the curve ξ gives also a ruling of the bundle $\rho : V \to L$ and $\xi \in |\rho^* O_L(1)|$ and the curve ε induces a ρ-ample line bundle $O_{V/L}(1)$. Their intersection numbers are given by $\xi^2 = 0$, $\xi.\varepsilon = 1$, and $\varepsilon^2 = -1$. The curve X which is a strict transform of X_0 is linearly equivalent to $5\xi + 3\varepsilon$. Then the restriction of the morphism $\rho|_X : X \to L$ gives a linear system g_3^1. A rational map $q : Y \cdots \to P = \mathbb{P}^4(\mathbb{C})$ induced by all the quadric curves passing through the point p_0 is lifted to a morphism $i : V \to P$, which gives an embedding of V into P. The very ample line bundle $O_V(H) = i^* O_P(1)$ has a member $2\xi+\varepsilon$. Since $K_V \sim -3\xi-2\varepsilon$, we see that $K_X \sim (K_V+X)|_X = (2\xi+\varepsilon)|_X$, which shows that the restriction of the embedding $j := i|_X X \to P$ is the canonical embedding of the curve X. The surface V has $deg(V) = H^2 = (2\xi + \varepsilon)^2 = 3$ and is a surface of minimal degree, or equivalently $\Delta(V, O_V(H)) = \dim V + deg(V) - h^0(V, O_V(H)) = 0$, which implies that the surface V is a homological shell of the curve X. The surface V is defined by 3 quadric equations and $h^0(P, I_X(2)) = 3$, which implies that the surface V is a quadric hull of X. If we have another surface V' of $deg(V') = 3$ which includes the curve X, is irreducible and reduced, and may have singularities, then the surface V' is a surface of $\Delta(V', O_{V'}(1)) = 0$ (cf. the structure theorem on the varieties of $\Delta = 0$, [1]), and is also defined by 3 quadric equations in $H^0(P, I_X(2))$, which shows $V' = V$. Thus, the surface V is a unique cubic surface including the curve X.

Theorem 2.1 *For the trigonal canonical curve of genus 5: $X \subseteq P = \mathbb{P}^4(\mathbb{C})$ and the unique cubic surface $V \subseteq P$ including the curve X, which is obtained from X through the process described above, we have the following table of the dimensions of cohomologies of several sheaves induced from the tangent sheaves $\Theta_\Box$ for $\Box = P, V, X, L, V/L$ and from the normal sheaves $N_\Box$ for $\Box = V, X, X/V$. We put $N_{(X,V)} := N_X \times_{\{N_V \otimes O_X\}} N_V$, which is the fiber product of N_X and N_V over $N_X \otimes O_X$ in the abelian category of coherent sheaves on P.*

sheaves	h^0	h^1	h^2	*sheaves*	h^0	h^1	h^2
$\rho^*\Theta_L$	*3*	*0*	*0*	$\Theta_P \otimes I_{X/V}$	*0*	*0*	*0*
$\Theta_{V/L}$	*3*	*0*	*0*	$\Theta_V \otimes I_{X/V}$	*0*	*0*	*1*
Θ_V	*6*	*0*	*0*	N_V	*18*	*0*	*0*
Θ_P	*24*	*0*	*0*	N_X	*36*	*0*	*0*
Θ_X	*0*	*12*	*0*	$N_{X/V}$	*17*	*0*	*0*
$\Theta_P \otimes O_V$	*24*	*0*	*0*	$N_V \otimes O_X$	*19*	*0*	*0*
$\Theta_P \otimes O_X$	*24*	*0*	*0*	$N_V \otimes I_{X/V}$	*0*	*1*	*0*
$\Theta_V \otimes O_X$	*6*	*1*	*0*	$N_{(X,V)}$	*35*	*0*	*0*

Table 1: The dimensions of cohomologies of several sheaves

The curve X and the surface V have the Hilbert polynomials $A_X(m) = 8m - 4$ and $A_V(m) = \frac{1}{2}(3m^2 + 5m + 2)$, respectively.

Proof. We show the rough outline of calculation on the data in the Table 1, which is rather annoying but a routine work in Algebraic Geometry. Applying Leray spectral sequence to $\rho^*\Theta_L \cong \rho^* O_{\mathbb{P}^1}(2) \cong O_V(2\xi)$, we get the row of $\rho^*\Theta_L$ in the table. Using the exact sequence :

$$0 \to \Theta_{V/L} \to \Theta_V \to \rho^*\Theta_L \to 0, \qquad (\#\text{-}4)$$

we see that $3\xi + 2\varepsilon \sim -K_V \sim det\Theta_V = \Theta_{V/L} \otimes \rho^*\Theta_L$ and $\Theta_{V/L} \cong O_V(\xi + 2\varepsilon)$. This brings the rows of $\Theta_{V/L}$ and of Θ_V in the table. Tensoring the ideal sheaf $I_{X/V} \cong O_V(-5\xi - 3\varepsilon)$ to the sequence (#-4), we see also the row of $\Theta_V \otimes I_{X/V}$. Recalling the Euler sequence :

$$0 \longrightarrow O_P \xrightarrow[\alpha_{EU}]{} \bigoplus_{j=0}^4 O_P(1)[Z_j] \xrightarrow[\beta_{EU}]{} \Theta_P \longrightarrow 0, \qquad (\#\text{-}5)$$

where $\{[Z_j]\}_{j=0}^4$ denotes the free base corresponding to the homogeneous coordinates $[Z_0 : \cdots : Z_4]$ of P, we see the row of Θ_P. To compute the rows of $\Theta_P \otimes O_V$ and of $\Theta_P \otimes O_X$, we make the tensor products with the sequence (#-5) and O_V or O_X. Then, we compare their long cohomology exact sequences and use the facts that $\alpha_{EU} = \Sigma Z_j[Z_j]$, the surface V and the curve X are linearly non-degenerate arithmetically Cohen-Macaulay closed subvarieties of P. For example, we have $H^1(O_V(m)) = 0$ and the surjectivities of the natural maps $H^0(O_P(m)) \to H^0(O_V(m))$ and $H^0(O_P(m)) \to H^0(O_X(m))$ and $H^0(O_P(1)) \cong H^0(O_V(1)) \cong H^0(O_X(1))$. In case of $\Theta_P \otimes O_V$, we use $O_V(1) \cong O_V(2\xi+\varepsilon)$, $K_V \sim -3\xi - 2\varepsilon$ and Serre duality and get the row of $\Theta_P \otimes O_V$ in the table. For $\Theta_P \otimes O_X$, we also use Serre duality, the explicit description of α_{EU}, and $O_X(1) \cong O_X(K_X)$ and get the row of $\Theta_P \otimes O_X$. The row of $\Theta_X \cong O_X(-K_X)$ is easy. After tensoring Θ_P to the sequence :

$$0 \to I_{X/V} \to O_V \to O_X \to 0, \qquad (\#\text{-}6)$$

the induced long cohomology exact sequence and the data on $\Theta_P \otimes O_V$ and on $\Theta_P \otimes O_X$ show the row of $\Theta_P \otimes I_{X/V}$. Similarly, by tensoring Θ_V to the sequence (#-6), the induced long cohomology exact sequence and the data on Θ_V and on $\Theta_V \otimes I_{X/V}$ bring the row of $\Theta_V \otimes O_X$. The data on N_V is obtained from the sequence :

$$0 \to \Theta_V \to \Theta_P \otimes O_V \to N_V \to 0. \qquad (\#\text{-}7)$$

Similarly, the data on N_X is easy to get. Tensoring $O_V(5\xi + 3\varepsilon)$ to the sequence (#-6), the isomorphism $N_{X/V} \cong O_V(5\xi + 3\varepsilon) \otimes O_X$ shows the data on $N_{X/V}$ in the table. We make a 3×3 exact commutative diagram by tensoring two short exact sequences (#-6) and (#-7), consider the big exact commutative diagram consisting from their long cohomology exact sequences, and obtain the data on $N_V \otimes O_X$ and on $N_V \otimes I_{X/V}$. The construction of the sheaf $N_{(X,V)}$ gives an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_V \otimes I_{X/V} & \longrightarrow & N_V & \longrightarrow & N_V \otimes O_X & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_V \otimes I_{X/V} & \longrightarrow & N_{(X,V)} & \longrightarrow & N_X & \longrightarrow & 0, \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N_{X/V} & = & N_{X/V} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array} \qquad (\#\text{-}8)$$

which brings the data on $N_{(X,V)}$. ∎

Under the circumstances of Theorem 2.1, let us consider the Hilbert scheme $\mathcal{H} = Hilb_P^{A_X(m)}$ and the flag Hilbert scheme $\mathcal{F} = FHilb_P^{A_X(m),A_V(m)}$ of $P = \mathbb{P}^4(\mathbb{C})$, which parametrize all the closed subschemes of P with the Hilbert polynomial $A_X(m)$, and all the pair of two closed subschemes of P with inclusions and a pair of the Hilbert polynomials $(A_X(m), A_V(m))$, respectively. Namely,

$$\mathcal{H}^{(cl.)} = \{[Y] \mid Y \subseteq P \ closed\ subscheme\ s.t.\ \chi(O_Y(m)) = A_X(m)\},$$

$$\mathcal{F}^{(cl.)} = \{([Y],[W]) \mid Y \subseteq W \subseteq P \ closed\ subschemes\ s.t.\ \chi(O_Y(m)) = A_X(m),\ \chi(O_W(m)) = A_V(m)\},$$

where $[Y]$, $[W]$ denote the closed points corresponding to the schemes Y and W, respectively. Since the scheme $\mathcal{F}$ is a closed subscheme of the product scheme $\mathcal{H} \times Hilb_P^{A_V(m)}$, the first projection morphism $pr_1 : \mathcal{H} \times Hilb_P^{A_V(m)} \to \mathcal{H}$ induces a proper morphism $\psi : \mathcal{F} \to \mathcal{H}$ which brings the map of closed points $\psi^{(cl.)} : \mathcal{F}^{(cl.)} \ni ([Y],[W]) \mapsto [Y] \in \mathcal{H}^{(cl.)}$. Obviously, for the trigonal canonical curve $X \subseteq P$ of genus 5 and the cubic surface V in Theorem 2.1, the corresponding closed points $[X]$ and $([X],[V])$ satisfy $[X] \in \mathcal{H}^{(cl.)}$ and $([X],[V]) \in \mathcal{F}^{(cl.)}$, respectively. From Table 1 in Theorem 2.1, The data $h^0(N_X) = 36$ and $h^1(N_X) = 0$ imply that the component of $\mathcal{H}$ is smooth at the closed point $[X]$ and of dimension 36. Similarly, the data $h^0(N_{(X,V)}) = 35$ and $h^1(N_{(X,V)}) = 0$ imply that the component of $\mathcal{F}$ is smooth at the closed point $([X],[V])$ and of dimension 35. Hence we choose uniquely an irreducible component $\mathcal{F}_0$ of $\mathcal{F}$ and an irreducible component $\mathcal{H}_0$ of $\mathcal{H}$ such that $([X],[V]) \in \mathcal{F}_0$, $[X] \in \mathcal{H}_0$, and $\psi(\mathcal{F}_0) \subseteq \mathcal{H}_0$. Now we set $\psi_0 := \psi|_{\mathcal{F}_0}$. Let us study the differential of the morphism $(d\psi_0)_{([X],[V])}$. Remind the facts on the tangent space that $\Theta_{\mathcal{F}_0,([X],[V])} \cong H^0(N_{(X,V)})$ and $\Theta_{\mathcal{H}_0,[X]} \cong H^0(N_X)$. Recalling the exact commutative diagram (#-8) and applying the data of Table 1, we have an exact commutative diagram of cohomology groups :

$$\begin{array}{ccccccccc}
0 = H^0(N_V \otimes I_{X/V}) & \longrightarrow & H^0(N_V)^{(18)} & \xrightarrow{\beta_{X/V}} & H^0(N_V \otimes O_X)^{(19)} & \xrightarrow[(surj.)]{\delta_{X/V}} & H^1(N_V \otimes I_{X/V})^{(1)} \\
 & & (surj.)\uparrow & & (surj.)\uparrow r & & \| \\
0 = H^0(N_V \otimes I_{X/V}) & \longrightarrow & H^0(N_{(X,V)})^{(35)} & \xrightarrow[d(\psi_0)_{([X],[V])}]{} & H^0(N_X)^{(36)} & \xrightarrow[\tau]{(surj.)} & H^1(N_V \otimes I_{X/V})^{(1)},
\end{array} \qquad (\#\text{-}9)$$

where the numbers in brackets appearing as superscripts in the right hand side denote the dimensions of the cohomology groups. This shows the differentials of the morphism $d(\psi_0)_{([X],[V])}$ is injective, which implies that the morphism ψ_0 maps $\mathcal{F}_0$ into $\mathcal{H}_0$ and is an embedding locally around the point $([X],[V])$. Thus the set $\mathcal{D} = \psi_0(\mathcal{F})$ is a Zariski closed subset of codimension 1 in $\mathcal{H}_0$. However, it still remains a possibility of difference on the tangent spaces : $d(\psi_0)_{([X],[V])}(\Theta_{\mathcal{F}_0,([X],[V])}) \neq \Theta_{\mathcal{D},[X]}$ (cf. Remark 2.2).

Remark 2.2 *Since the scheme $\mathcal{H}_0$ is smooth at the point $[X]$, the "divisor" (or more precisely, the codimension 1 closed subscheme of $\mathcal{H}_0$ with the reduced structure) D is a Cartier divisor locally around the point $[X]$. However, we still can not claim that the scheme $\mathcal{D}$ is smooth around the point $[X]$. Namely, there might exist another different closed point $([X],[W]) \in \mathcal{F}_0$ with $A_W(m) = A_V(m)$. We already exclude the case that W is irreducible and reduced. However, for example, we can not yet exclude the case that $W = W_0 \cup Y$ by a primary decomposition, where W_0 is of purely dimension 2, $deg(W_0) = 3$, $X \not\subseteq W_0$, $X \subseteq Y$, and $\dim Y \leq 1$.*

In spite of the problem pointed out in the Remark 2.2, we can roughly consider the short exact sequence obtained in (#-9) as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(N_{(X,V)})^{(35)} & \xrightarrow[d(\psi_0)_{([X],[V])}]{} & H^0(N_X)^{(36)} & \xrightarrow[\tau]{(surj.)} & H^1(N_V \otimes I_{X/V})^{(1)} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & “\Theta_{\mathcal{D},[X]}” & \longrightarrow & \Theta_{\mathcal{H},[X]} & \longrightarrow & N_{\mathcal{D}/\mathcal{H},[X]} & \longrightarrow & 0 \end{array} \tag{\#-10}$$

Since the closed point $[X]$ is a smooth point of $\mathcal{H}$, we can take a smooth affine curve B in $\mathcal{H}$ passing through the point $[X]$ and "crossing $\mathcal{D}$ transversely" and set theoretically $B \cap \mathcal{D} = \{[X]\}$. Setting $b_0 = [X]$, we obtain a flat family $f : \mathfrak{X} \to B$ of canonical curve of genus 5 in $P = \mathbb{P}^4(\mathbb{C})$. From an intuition arising from the sequence (#-10), as the transverseality condition on the intersection of B and $\mathcal{D}$, we assume that for the section $\sigma \in H^0(N_X)$ which gives the tangent direction of the curve B, $\tau(\sigma) \neq 0 \in H^1(N_V \otimes I_{X/V})$.

From the classical works [5], [6], [7], we see that for a closed point $b \neq b_0 \in B^{(cl.)}$, the curve $X(b)$ is a non-trigonal curve, the minimal graded S-free resolution of the homogeneous coordinate ring $R_{X(b)}$ has the form :

$$0 \leftarrow R_{X(b)} \leftarrow S \leftarrow S(-2)^{\oplus 3} \leftarrow S(-4)^{\oplus 3} \leftarrow S(-6) \leftarrow 0, \tag{\#-11}$$

and for the point b_0, the curve $X(b_0) = X$ is a trigonal curve, the minimal graded S-free resolution of the homogeneous coordinate ring $R_{X(b_0)}$ has the form :

$$0 \leftarrow R_{X(b_0)} \leftarrow S \leftarrow S(-2)^{\oplus 3} \oplus \underline{S(-3)^{\oplus 2}} \leftarrow S(-4)^{\oplus 3} \oplus \underline{S(-3)^{\oplus 2}} \leftarrow S(-6) \leftarrow 0. \tag{\#-12}$$

Comparing two minimal graded S-free resolutions (#-11) and (#-12), the degeneration of syzygies occur at the first syzygy in degree 3 and the second syzygy in degree 3, which means that the family $f : \mathfrak{X} \to B$ is 0-Betti constant but is not 1-Betti constant. By Theorem 1.7, the degeneration of the first syzygy in this case is in our researchable range. Thus, our main problem is to determine the module structure of the O_B-module $\mathscr{T}_3^{1,1}$.

Now we have a table which describes the conditions on the stalks $\{\mathscr{T}_{3,b_0}^{p,q}\}$ $(p = 0, 1)$ and the spaces $T_3^{1,q}(b)$ $(b = b_0$ or $= b' \neq b_0)$ relating with studying the module $\mathscr{T}_{3,b_0}^{1,1}$.

q	0	1	2	3	4	5
$\dim T_3^{1,q}(b)\ \ (b'/b_0)$	0	0/2	0/2	0	0	0
$(C.B.C.)_3^{1,q}(b_0)$	o.k.	o.k.	NO	o.k.	o.k.	o.k.
$(C.B.C.)_3^{0,q}(b_0)$	o.k.	NO	o.k.	o.k.	o.k.	o.k.
$(L.F.)_3^{1,q}(b_0)$	o.k. $\{0\}$	NO (torsion)	o.k. $\{0\}$	o.k. $\{0\}$	o.k. $\{0\}$	o.k. $\{0\}$
$(L.F.)_3^{0,q}(b_0)$	o.k.	o.k. $\{0\}$	o.k.	o.k.	o.k.	o.k.

Table 2: The conditions on the stalks $\mathscr{T}_{3,b_0}^{p,q}$ and the spaces $T_3^{1,q}(b)$

In Table 2 above, the items without an underline are obvious to see. Since the local ring O_{B,b_0} is a discrete valuation ring, its global dimension is finite, which implies that the homological dimension of the stalk $\mathscr{T}_{m,b_0}^{1,q}$ at the point b_0 satisfies $hd_{O_{B,b_0}}(\mathscr{T}_{m,b_0}^{1,q}) \leq 1$ by Auslander-Buchsbaum formula. From the 0-Betti constancy of the family $f : \mathfrak{X} \to B$, we have an exact sequence of the stalks at any closed point $b \in B^{(cl.)}$:

$$0 \to \mathscr{T}_{m,b}^{0,q} \to \oplus\, \mathscr{T}_{m-q,b}^{0,0} \to \mathscr{T}_{m,b}^{0,q-1} \to \mathscr{T}_{m,b}^{1,q} \to 0. \qquad (\#\text{-}13)$$

For $b = b_0$ and $m = 3$, applying the claim (1.7.1) and starting from the case $q = 1$, the exact sequence (#-13) above shows the conditions $(L.F.)_3^{0,q}(b_0)$ $(q = 1, 2)$ hold. As we see in the sequel, $\mathscr{T}_{3,b}^{1,1} = 0$ for $b \neq b_0$, which implies $\mathscr{T}_{3,b}^{0,1} = 0$ for $b \neq b_0$ after counting the ranks of $\mathscr{T}_{2,b}^{0,0}$ and $\mathscr{T}_{3,b}^{0,0}$ for $b \neq b_0$. Hence, the freeness of the module $\mathscr{T}_{3,b_0}^{0,1}$ implies that the coherent sheaf $\mathscr{T}_3^{0,1}$ is zero. Then, the sequence (#-13) for $b = b_0$, $m = 3$ and $q = 2$ shows $\mathscr{T}_{3,b_0}^{1,2} = 0$. By Theorem 12.11 (b) in Chap. III in [3], we see that the condition $(C.B.C.)_3^{0,1}(b_0)$ does not hold because the condition $(L.F.)_3^{1,1}(b_0)$ does not hold. Then, Proposition 1.8 implies that $(C.B.C.)_3^{1,2}(b_0)$ does not hold.

Remark 2.3 *As a result from Table 2, we can find a typical example related to syzygies where the cohomological base change property does NOT hold :*

$$\varphi_3^{1,2}(b_0) : \mathscr{T}_3^{1,2} \otimes k(b_0) = 0 \overset{not\ surj.}{\longrightarrow} T_3^{1,2}(b_0) \neq 0.$$

This phenomenon brings the following observation to us. Since the family $f : \mathfrak{X} \to B$ of our example in study is 0-Betti constant and not 1-Betti constant, we can detect only the degeneration of the first syzygies and not those of higher syzygies by our technical limitation (cf. Theorem 1.7). Here we can determine by chance the module structure of the sheaf $\mathscr{T}_3^{1,2}$ which locates out of our technical limitation. This result shows that in this case, the sheaf $\mathscr{T}_3^{1,2}$ does not reflect the degeneration of the second syzygies from the begining. To find out a new tool for getting over our technical limitation, it may be interesting to determine the module structure of $\overline{\mathscr{T}}_{3,\nu}^{1,2}$ instead of the sheaf $\mathscr{T}_3^{1,2}$.

Now we consider the module structure of the stalk $\mathscr{T}_{3,b_0}^{1,1}$. By Theorem 1.7, for any closed point $b \in B^{(cl.)}$, we see that $\mathscr{T}_3^{1,1} \otimes k(b) \cong T_3^{1,1}(b)$. If $b \neq b_0$, $T_3^{1,1}(b) = 0$, we see that the module $\mathscr{T}_3^{1,1}$ is a torsion O_B-module and $Supp(\mathscr{T}_3^{1,1}) = \{b_0\}$. Moreover, $T_3^{1,1}(b_0) \cong \mathbb{C}^2$, we see that $\mathscr{T}_3^{1,1} \cong O_{B,b_0}/(t^{k_1}) \oplus O_{B,b_0}/(t^{k_2})$, where " t " denotes the regular parameter of the discrete valuation ring O_{B,b_0}. Now we have a conjecture as follows.

Conjecture 2.4 *If $\tau(\sigma) \neq 0 \in H^1(N_V \otimes I_{X/V})$, then $k_1 = k_2 = 1$ (?).*

Let us recall our results [13]. From the first row in the diagram (#-3) in [13] with tensoring $\Omega^1_{P\times\overline{B}_1/\overline{B}_1}(3)$, we have a long exact sequence :

$$0 \to T_3^{0,1}(b_0) \to (\overline{\mathscr{T}}^{0,1}_{3,1})_{b_0} \xrightarrow{\lambda} T_3^{0,1}(b_0) \xrightarrow{ob_\sigma} T_3^{1,1}(b_0) \xrightarrow{\mu} (\overline{\mathscr{T}}^{1,1}_{3,1})_{b_0} \to T_3^{1,1}(b_0) \to 0, \qquad (\#\text{-}14)$$

where the map ob_σ is the same as the map $\delta^{(0)}_{IDF}$ in [13], and the final surjectivity in the sequence (#-14) comes from the surjectivity of $(\mathscr{T}_3^{1,1})_{b_0} \to T_3^{1,1}(b_0)$ with respect to the cohomological base change at b_0, which is factored naturally into $(\mathscr{T}_3^{1,1})_{b_0} \to (\overline{\mathscr{T}}^{1,1}_{3,1})_{b_0} \to T_3^{1,1}(b_0)$. By Theorem 1.4, on Conjecture 2.4, the conclusion part $k_1 = k_2 = 1$ holds if and only if the map $\mu = 0$, which is equivalent to the surjectivity of ob_σ. On the other hand, from the argument above, we see that the condition $(C.B.C)_3^{1,1}(b)$ holds for any $b \in B^{(cl.)}$ and $(L.F.)_3^{1,1}(b_0)$ does not hold, the condition $(C.B.C)_3^{0,1}(b_0)$ does not hold, namely the natural map $(\mathscr{T}_3^{0,1})_{b_0} \to T_3^{0,1}(b_0)$ is not surjective. Thus we have a chance that the map λ does not have the surjectivity, which is almost an obstruction for the surjectivity of the map ob_σ. For example, if the module $(\overline{\mathscr{T}}^{1,1}_{3,1})_{b_0}$ is $O_{\overline{B}_1,b_0}$-locally free, then the map λ is surjective and the map ob_σ is zero, which is far from the surjectivity.

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