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# Betti constancy of the flat families of projective subschemes over non-reduced schemes

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**Abstract**

To study more precisely the tangent spaces of loci in the Hilbert schemes of the complex projective $N$-space $P = \mathbb{P}^N(\mathbb{C})$ where the graded Betti numbers of the closed subschemes of $P$ are preserved, we improve our results in the previous article [7] for removing the conditions on generalized graded Betti numbers.

**Keywords**: Hilbert scheme, tangent space, graded Betti number

## §1 Introduction.

Let us consider the Hilbert scheme $\mathcal{H} = Hilb^{A(m)}$ which parametrizes the closed subschemes in $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ with a Hilbert polynomial $A(m)$ and its universal family $\varphi : \mathcal{U} \to \mathcal{H}$ over $\mathcal{H}$. For a closed point $x \in \mathcal{H}^{(cl.)}$, its fiber $X = \varphi^{-1}(x)$ is a closed subscheme of $P$, and this closed point $x$ is denoted also by $[X]$. By taking the graded Betti numbers $\{b_{q,m}(X)\}_{q,m}$ of $X$, or those of the homogeneous coordinate ring $R_X$ of $X$ as a graded $S$-module, we have functions $b_{q,m} : \mathcal{H}^{(cl.)} \to \mathbb{Z}_{\geq 0}$ $(q,\ m \geq 1)$. Contrary to a naive expectation, these functions do not have the upper semi-continuity in general. Now we choose the maximal Zariski open set $\mathcal{H}_0 \subseteq \mathcal{H}$ whose closed point $x \in \mathcal{H}_0^{(cl.)}$ has the fiber $X = \varphi^{-1}(x)$ satisfying the arithmetic $D_2$-condition : $H^1(I_X(m)) = 0$ $(\forall m \in \mathbb{Z})$. Restricting these functions $\{b_{q,m}\}_{q,m}$ to the open set $\mathcal{H}_0$, they have a cohomological expression: $\beta_{q,m}(X) = h^1(\Omega_P^q \otimes I_X(m))$ $(q,\ m \geq 1)$ and therefore have the upper semi-continuity.

Now we take an arithmetically $D_2$ closed subscheme $X$ of $P$ with the Hilbert polynomial $A(m)$. Then we define "the Betti constant set of $X$" $BC(X)^{(set)} := \{y \in \mathcal{H}_0^{(cl.)} \mid b_{q,m}(\varphi^{-1}(y)) = b_{q,m}(X) \quad (\forall q, m \in \mathbb{Z}_{\geq 1})\}$, which is a locally closed set in $\mathcal{H}_0^{(cl.)}$ and obviously $[X] \in BC(X)^{(set)}$. We can also construct a locally closed subscheme $BC(X)$ of $\mathcal{H}_0$ which possesses the universality, and consider the set $BC(X)^{(set)}$ as the set of closed points $BC(X)^{(cl.)}$ of the scheme $BC(X)$(cf. [8]). This scheme $BC(X)$ is called as "the Betti constant scheme of $X$", or "the Betti constant locus of $X$". By the theory of Hilbert schemes (cf. [1], [4], [6]), the tangent space $T_{\mathcal{H},[X]}$ of the Hilbert scheme $\mathcal{H}$ at the closed point $[X] \in \mathcal{H}_0^{(cl.)}$ is identified with the space of global sections $H^0(N_X)$ of the normal sheaf $N_X$ of $X$ in $P$.

One of the aims of this article is to present a criterion for a given global normal vector field $\sigma \in H^0(N_X) \cong T_{\mathcal{H},[X]}$ being included in the tangent space $T_{BC(X),[X]}$ of the Betti constant locus $BC(X)$.

For that aim, we have to study generally projective and flat families whose fibres are arithmetically $D_2$ and preserving the graded Betti numbers over algebraic schemes which may be non-reduced, which

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include the spectrum of the ring of dual numbers. The key point of our study is to give an appropriate definition of "Betti constancy" which can work also on non-reduced base schemes and admits to construct the universal families, namely a good stratification of the arithmetically $D_2$ locus $\mathcal{H}_0$ in the Hilbert scheme $\mathcal{H}$. In particular, we have to give a guarantee on the cohomological base change properties of the several higher direct image sheaves on non-reduced base schemes. Since it was not so easy to show in general the cohomological base change properties on non-reduced base schemes, in our previous article [7], we introduced the new concepts "generalized graded Betti numbers" and "strong Betti constancy" and then, handled all the higher direct image sheaves including the parts which do not correspond directly to the (ordinary) graded Betti numbers. In this article, we improve our method of proof in the previous article [7], remove the conditions on generalized graded Betti numbers, or strong Betti constancy, and handle exactly the Betti constant loci.

Also in this article, we again refer to [3] Chap. III §12, to [2] Chap. III §7, and to [5] Lect. 7 as the useful general results on cohomological bease change.

# §2 Results.

All the objects and the morphisms considered in this article are algebraic schemes over the complex number field $\mathbb{C}$ and morphisms of finite type. We investigate our objects in the following common circumstances.

**Circumstances 2.1** *Let $B$ a connected algebraic scheme over $\mathbb{C}$ (which may be non-reduced), $P = \mathbb{P}^N(\mathbb{C})$, and $f = \pi|_{\mathfrak{X}} : \mathfrak{X} \to B$ a projective and flat morphism as in the following figure (#-1), where $\mathfrak{X}$ and "incl." denote a closed subscheme of $P \times B$ and an inclusion morphism, respectively.*

*We assume that for any closed point $b \in B^{(cl.)}$, the fiber $X(b) = f^{-1}(b)$ is an arithmetically $D_2$ closed subscheme of $P$ (N.B. hence $R^1\pi_*(I_{\mathfrak{X}}(m)) = 0$ for any integer $m \in \mathbb{Z}$ including the case $m < 0$).*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{incl.} & P \times B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi = pr_B} \\ & & B \end{array} \qquad (\#\text{-}1)$$

*For any point $b \in B$ and for all the integers $p$, $q$, $m$ with $p \geq 0$, $q \geq 0$, the natural maps for cohomological base change (with respect to the morphism $\pi$) of the sheaf $\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}}$ on $P \times B$ are written as $\varphi^{p,q}_m(b) : R^p\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}}) \otimes k(b) \to H^p(\Omega^q_P(m) \otimes I_{X(b)})$.*

Let us introduce our key concept of "Betti constancy" for the families over non-reduced schemes.

**Definition 2.2 (Betti constant family)** *Under Circumstances 2.1, we say that the family $f : \mathfrak{X} \to B$ is* Betti constant *if the coherent sheaves $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ are $O_B$-locally free sheaves for all the integers $q$, $m \geq 1$.*

For Main Theorem 2.5 in the sequel, we prepare the next lemma.

**Lemma 2.3** *Under Circumstances 2.1, we have two properties on the sheaves $\pi_*(I_{\mathfrak{X}}(m))$ ($\forall m \in \mathbb{Z}$).*

(2.3.1) *The coherent sheaves $\pi_*(I_{\mathfrak{X}}(m))$ are $O_B$-locally free sheaves for all the integers $m$.*

(2.3.2) *For any closed point* $b \in B^{(cl.)}$ *and for any integer* $m$*, the natural map* $\varphi_m^{0,0}(b) : \pi_*(I_{\mathfrak{X}}(m)) \otimes k(b) \to H^0(I_{X(b)}(m))$ *is an isomorphism.*

**Proof.** Since $R^1\pi_*(I_{\mathfrak{X}}(m)) = 0$ and $H^1(I_{X(b)}(m)) = 0$ for any closed point $b \in B^{(cl.)}$ and for any $m \in \mathbb{Z}$ with including the case $m < 0$. Then Theorem 12.11 (b) and (a) in [3] show that we have an isomorphism :

$$\varphi_m^{0,0}(b) : \pi_*(I_{\mathfrak{X}}(m)) \otimes k(b) \to H^0(I_{X(b)}(m)).$$

Since $\pi_*(O_{P\times B}(m)) \cong H^0(O_P(m)) \otimes O_B$, for any closed point $b \in B^{(cl.)}$ and for any $m \in \mathbb{Z}$, we obtain an exact commutative diagram :

$$\begin{array}{ccccccccc}
 & & \pi_*(I_{\mathfrak{X}}(m)) \otimes k(b) & \xrightarrow[\alpha_{ST}(b)]{} & \pi_*(O_{P\times B}(m)) \otimes k(b) & \xrightarrow[\beta_{ST}(b)]{} & \pi_*(O_{\mathfrak{X}}(m)) \otimes k(b) & \longrightarrow & 0 \\
 & & {\scriptstyle \varphi_m^{0,0}(b)}\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & H^0(I_{X(b)}(m)) & \longrightarrow & H^0(O_P(m)) & \longrightarrow & H^0(O_{X(b)}(m)) & \longrightarrow & 0,
\end{array}$$

which shows the injectivity of the homomorphism $\alpha_{ST}(b)$, and therefore $Tor_1^{O_{B,b}}(\pi_*(O_{\mathfrak{X}}(m))_b, k(b)) = 0$ for the stalk $\pi_*(O_{\mathfrak{X}}(m))_b$ at the closed point $b$. Hence we see that the stalk $\pi_*(O_{\mathfrak{X}}(m))_b$ is $O_{B,b}$-free, which implies the sheaf $\pi_*(O_{\mathfrak{X}}(m))$ is $O_B$-locally free. Thus we see that the sheaf $\pi_*(I_{\mathfrak{X}}(m))$ is also $O_B$-locally free. ∎

**Remark 2.4** *Under Circumstances 2.1, the arithmetic* $D_2$*-condition implies that every closed fiber has positive dimension, and is connected.*

In the previous article [7], to guarantee the cohomological base change properties of the sheaves $R^1\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ on the non-reduced base scheme $B$ by using the sheaves $R^p\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ $(p \geq 2)$, we introduced two kinds of concepts, "generalized graded Betti numbers" and "strongly Betti constant families".

Now we can prove the cohomological base change properties of the sheaves $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ on the non-reduced base scheme $B$ without the aide of the sheaves $R^p\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ $(p \geq 2)$.

**Main Theorem 2.5** *Under Circumstances 2.1, we suppose that the family* $f : \mathfrak{X} \to B$ *is Betti constant, namely the coherent sheaves* $R^1\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ *are* $O_B$*-locally free sheaves for all the integers* $q$*,* $m \geq 1$*. Then we have the following two properties.*

(2.5.1) *The coherent sheaves* $\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ *are* $O_B$*-locally free sheaves for all the integers* $q \geq 0$*,* $m \geq 1$*.*

(2.5.2) *If* $p = 0$ *or* $p = 1$*, for any closed point* $b \in B^{(cl.)}$ *and for all the integers* $q \geq 0$*,* $m \geq 1$*, the natural map* $\varphi_m^{p,q}(b) : R^p\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}}) \otimes k(b) \to H^p(\Omega^q_P(m) \otimes I_{X(b)})$ *is an isomorphism.*

**Proof.** To prove this Main Theorem, we will apply an induction on $q$. In Lemma 2.3, we already proved both of the two assertions above for the case $q = 0$ in the stronger form. Let us set $q \geq 1$ and assume that both of the two assertions (2.5.1) and (2.5.2) hold for the case $q - 1$.

Now we take a closed point $b \in B^{(cl.)}$, and apply the functors $R^\bullet\pi_*(-)$ and $H^\bullet(P,-)$to an exact sequence of sheaves on $P \times B$ arising from the $q$-th exterior product of the Euler sequence of the sheaf $\Omega^1_{P\times B/B}$:

$$0 \longrightarrow \Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}} \longrightarrow \oplus I_{\mathfrak{X}}(m-q) \longrightarrow \Omega^{q-1}_{P\times B/B}(m)\otimes I_{\mathfrak{X}} \longrightarrow 0, \qquad (\#\text{-}2)$$

and a similar exact sequence on $P$:

$$0 \longrightarrow \Omega^q_P(m)\otimes I_{X(b)} \longrightarrow \oplus I_{X(b)}(m-q) \longrightarrow \Omega^{q-1}_P(m)\otimes I_{X(b)} \longrightarrow 0, \qquad (\#\text{-}3)$$

respectively, and obtain an exact sequence of sheaves:

$$\begin{aligned} 0 \longrightarrow\ & \pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}}) \longrightarrow \oplus\pi_*(I_{\mathfrak{X}}(m-q)) \longrightarrow \pi_*(\Omega^{q-1}_{P\times B/B}(m)\otimes I_{\mathfrak{X}}) \\ \longrightarrow\ & R^1\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}}) \longrightarrow 0, \end{aligned} \qquad (\#\text{-}4)$$

and an exact sequence of cohomologies :

$$\begin{aligned} 0 \longrightarrow\ & H^0(\Omega^q_P(m)\otimes I_{X(b)}) \longrightarrow \oplus H^0(I_{X(b)}(m-q)) \longrightarrow H^0(\Omega^{q-1}_P(m)\otimes I_{X(b)}) \\ \longrightarrow\ & H^1(\Omega^q_P(m)\otimes I_{X(b)}) \longrightarrow 0, \end{aligned} \qquad (\#\text{-}5)$$

where we use $R^1\pi*(I_{\mathfrak{X}}(m')) = 0$ and $H^1(I_{X(b)}(m')) = 0$ for any $m' \in \mathbb{Z}$. Then we put a sheaf $\underline{\mathcal{K}}^{q-1}_m$ to be

$$\underline{\mathcal{K}}^{q-1}_m := Ker[\pi_*(\Omega^{q-1}_{P\times B/B}(m)\otimes I_{\mathfrak{X}}) \to R^1\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})].$$

By the assumption of Betti constancy and the induction hypothesis, its definition implies that the sheaf $\underline{\mathcal{K}}^{q-1}_m$ is $O_B$-locally free, and therefore the sheaf $\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})$ is also $O_B$-locally free by Lemma 2.3. Hence we have an exact sequence after tensoring the residue field $k(b)$ at the point $b$ to the sequence (#-4) and obtain an exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})\otimes k(b) & \longrightarrow & \oplus\pi_*(I_{\mathfrak{X}}(m-q))\otimes k(b) & \xrightarrow{u} & \pi_*(\Omega^{q-1}_{P\times B/B}(m)\otimes I_{\mathfrak{X}})\otimes k(b) \\ & \varphi^{0,q}_m(b) & & \cong\ \oplus\varphi^{0,0}_{m-q}(b) & & \cong\ \varphi^{0,q-1}_m(b) \\ 0 \longrightarrow & H^0(\Omega^q_P(m)\otimes I_{X(b)}) & \longrightarrow & \oplus H^0(I_{X(b)}(m-q)) & \xrightarrow{v} & H^0(\Omega^{q-1}_P(m)\otimes I_{X(b)}) \end{array}$$

$$\begin{array}{ccc} \longrightarrow & R^1\pi_*(\Omega^q_{P\times B/B}(m)\otimes I_{\mathfrak{X}})\otimes k(b) \longrightarrow & 0 \\ & \varphi^{1,q}_m(b) & \\ \longrightarrow & H^1(\Omega^q_P(m)\otimes I_{X(b)}) \longrightarrow & 0. \end{array}$$

(#-6)

By Lemma 2.3 and the induction hypothesis, we have isomorphisms $\oplus\varphi^{0,0}_{m-q}(b)$ and $\varphi^{0,q-1}_m(b)$. Then both of the homomorphisms $u$ and $v$ coincide with each other through these isomorphisms. Hence the kernels and the cokernels of the homomorphisms $u$ and $v$ coincide with each other through these isomorphisms. Thus we see that the homomorphisms $\varphi^{0,q}_m(b)$ and $\varphi^{1,q}_m(b)$ are isomorphisms. ∎

Let us recall our previous results of [7] in a summarized form.

**Theorem 2.6** *Under Circumstances 2.1, we set $B = T_\varepsilon = Spec\ \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Take an arithmetically $D_2$ closed subscheme $X \subseteq P = \mathbb{P}^N(\mathbb{C})$, a section $\sigma \in H^0(N_{X/P}) \cong H^0(\underline{Hom}_{O_P}(I_X, O_X))$, and the projective and flat family $f : \mathfrak{X} \to T_\varepsilon$ which corresponds bijectively to the section $\sigma$ and satisfies $f^{-1}(t_0) = X$ for the unique closed point $t_0 \in T_\varepsilon^{(cl.)}$. Fix integers $p,\ q,\ m \geq 1$. Then, a cohomology class $\tau \in H^p(\Omega^q_P(m) \otimes I_X)$ is included by the image of the natural homomorphism :*

$$\varphi^{p,q}_m(t_0) : R^p(\Omega^q_{P\times T_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \otimes k(t_0) \to H^p(\Omega^q_P(m) \otimes I_X)$$

*if and only if the class $\tau \cup \sigma \in H^p(\Omega^q_P(m) \otimes O_X)$ obtained by the natural coupling:*

$$H^p(\Omega^q_P(m) \otimes I_X) \times H^0(\underline{Hom}_{O_P}(I_X, O_X)) \to H^p(\Omega^q_P(m) \otimes O_X)$$

*is zero. In particular, the homomorphism $\varphi^{p,q}_m(t_0)$ is surjective if and only if the coupling homomorphism*

$$\sigma^{p,q}_m = \cup\sigma : H^p(\Omega^q_P(m) \otimes I_X) \to H^p(\Omega^q_P(m) \otimes O_X)$$

*is a zero map.*

**Proof.** To avoid taking many pages for explaining our notation on several cohomological operators, we refer to [7], in particular, Theorem 2.6 and Lemma 2.7 of [7]. ∎

**Remark 2.7** *In Theorem 2.6, the condition $p \geq 1$ is essential. If $p = 0$, then it is still true that $\tau\cup\sigma = 0$ implies $\tau \in Im(\varphi^{p,q}_m(t_0))$. However, the converse is not true in general for $p = 0$.*

Now we can give an improved version of Main Theorem 2.9 of [7].

**Theorem 2.8** *Let $X$ be an arithmetically $D_2$ closed subscheme of a projective space $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ whose Hilbert polynomial is $A(m)$. Take the Hilbert scheme $\mathcal{H} = Hilb^{A(m)}_P$ of $P$ which parametrizes all the closed subschemes of $P$ whose Hilbert polynomials coincide with $A(m)$. The tangent space $T_{\mathcal{H},[X]}$ of $\mathcal{H}$ at the closed point $[X]$ is naturally isomorphic to the space of global normal vector fields $H^0(N_{X/P})$ of $X$ in $P$. A section $\sigma \in H^0(N_{X/P}) \cong H^0(\underline{Hom}_{O_P}(I_X, O_X))$ corresponds bijectively to a projective and flat family $f : \mathfrak{X} = \mathfrak{X}_\sigma \to T_\varepsilon = Spec(\ \mathbb{C}[\varepsilon]/(\varepsilon^2))$ with $\mathfrak{X} \subseteq P \times T_\varepsilon$, $f = pr_{T_\varepsilon}|_{\mathfrak{X}}$, and $f^{-1}(t_0) = X$ for the unique closed point $t_0 \in T_\varepsilon^{(cl.)}$.*

*Then the family $f : \mathfrak{X} \to T_\varepsilon$ is Betti constant if and only if the followoing condition (*) holds.*

<u>Condition(*)</u>
*The natural map $\sigma^{1,q}_m = \cup\sigma : H^1(\Omega^q_P(m) \otimes I_X) \to H^1(\Omega^q_P(m) \otimes O_X)$ by the coupling with $\sigma \in H^0(N_{X/P})$ is a zero map $(\forall q, m \in \mathbb{Z}_{\geq 1})$.*

*In other words,*

$$\begin{array}{rcl} T_{BC(X),[X]} & = & \{\sigma \in H^0(N_{X/P}) \cong T_{\mathcal{H},[X]} \mid \sigma^{1,q}_m = 0 \text{ as the map} \\ & : & H^1(\Omega^q_P(m) \otimes I_X) \to H^1(\Omega^q_P(m) \otimes O_X) \quad (\forall q, m \in \mathbb{Z}_{\geq 1})\}. \end{array}$$

**Proof.** Let us assume that the family $f : \mathfrak{X} \to T_\varepsilon$ is Betti constant. Then Main Theorem 2.5 (2.5.2) and Theorem 2.6 show the Condition(*).

Conversely, we assume the condition (*). Following after the argument of Main Theorem 2.9 in [7], for fixed integers $p, q, m$, we introduce again two abbreviations on the conditions of locally freeness and of the cohomological base change for simplifying our proof.

$(L.F.)^{p,q}_m$ : $R^p\pi_*(\Omega^q_{P\times T_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})$ is an $O_{T_\varepsilon}$-locally free sheaf.

$(C.B.C.)^{p,q}_m$ : The natural map $\varphi^{p,q}_m(t_0) : R^p\pi_*(\Omega^q_{P\times T_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})\otimes k(t_0) \to H^p(\Omega^q_P(m)\otimes I_X)$ is surjective, or equivalently bijective (cf. Theorem 12.11 (a) in [3]).

The condition (*) is equivalent to the conditions $(C.B.C.)^{1,q}_m$ for $(\forall q,\ m \geq 1)$. What we have to show is the conditions $(L.F.)^{1,q}_m$ for $(\forall q,\ m \geq 1)$. By applying Theorem 12.11 (b) in [3], it is enough to show the conditions $(C.B.C.)^{0,q}_m$ for $(N \geq \forall q \geq 1,\ \forall m \geq 1)$.

If $q = N$, then $\pi_*(\Omega^N_{P\times T_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}}) \cong \pi_*(I_{\mathfrak{X}}(m-N-1))$, $H^0(\Omega^N_P(m)\otimes I_X) \cong H^0(I_X(m-N-1))$, and $\varphi^{0,N}_m(t_0) = \varphi^{0,0}_{m-N-1}(t_0)$ through these natural isomorphisms. Hence, Lemma 2.3 brings the condition $(C.B.C.)^{0,N}_m$ $(\forall m \geq 1)$. Then we may assume $q \leq N-1$. Let us recall the diagram (#-6) by replacing $X(b)$ by $X$, $B$ by $T_\varepsilon$, $b$ by $t_0$, and $q$ by $q+1$, respectively.

$$\begin{array}{ccccc}
\oplus\pi_*(I_{\mathfrak{X}}(m-q-1))\otimes k(t_0) & \longrightarrow & \pi_*(\Omega^q_{P\times T_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})\otimes k(t_0) & \xrightarrow[(surj.)]{\delta'} & R^1\pi_*(\Omega^{q+1}_{P\times T_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})\otimes k(t_0) \\
\cong\ \oplus\varphi^{0,0}_{m-q-1}(t_0) & & \varphi^{0,q}_m(t_0) & & \cong\ \varphi^{1,q+1}_m(t_0) \\
\oplus H^0(I_X(m-q-1)) & \longrightarrow & H^0(\Omega^q_P(m)\otimes I_X) & \xrightarrow[(surj.)]{\delta''} & H^1(\Omega^{q+1}_P(m)\otimes I_X).
\end{array}$$

(#-7)

The homomorphisms $\delta'$ and $\delta''$ in (#-7) are surjective. Since we have the condition $(C.B.C.)^{1,q+1}_m$, the homomorphism $\varphi^{1,q+1}_m(t_0)$ is an isomorphism. The homomorphism $\oplus\varphi^{0,0}_{m-q-1}(t_0)$ is an isomorphism by Lemma 2.3. By easy diagram chasing, we see that the homomorphism $\varphi^{0,q}_m(t_0)$ is surjective, namely the condition $(C.B.C.)^{0,q}_m$. ∎

**Corollary 2.9** *Under the circumstances of Theorem 2.8, we assume moreover that the closed subscheme $X$ satisfies the arithmetic $D_3$-condition and its graded Betti numbers satisfy : for any $q \geq 1$ and $m \geq 1$, if $b_{q,m}(X) > 0$, then $b_{q-1,m}(X) = 0$ e.g. the homogeneous coordinate ring $R_X$ of $X$ has a pure resolution of type $(d_1, d_2, \ldots, d_\ell)$ with $\ell \leq N-2$ (cf. Remark 2.10) as a graded minimal $S$-free resolution of $R_X$. Here is a remark that $b_{0,m} = h^1(\Omega^1(m)\otimes O_X)$ is 1 if $m = 0$, and 0 otherwise.*

*Then $T_{BC(X),[X]} = T_{\mathcal{H},[X]}$. Namely, at the infinitesimal neighborhood of the point $[X]$ of order 1, the graded Betti numbers of $X$ has constancy in all directions in the Hilbert scheme $\mathcal{H}$.*

**Proof.** Recall that $b_{q,m}(X) = h^1(\Omega^q(m)\otimes I_X)$ for any $q,\ m \geq 1$. The arithmetic $D_3$-condition of $X$ implies that $H^1(O_X(m)) = 0$ for any $m \in \mathbb{Z}$. Then, for any $q \geq 1$ and $m \geq 1$, we have the equality : $b_{q-1,m}(X) = h^1(\Omega^q(m)\otimes O_X)$. Thus, for any $\sigma \in H^0(N_{X/P}) \cong T_{\mathcal{H},[X]}$, the natural map $\sigma^{1,q}_m = \cup\sigma : H^1(\Omega^q_P(m)\otimes I_X) \to H^1(\Omega^q_P(m)\otimes O_X)$ is a zero map $(\forall q, m \in \mathbb{Z}_{\geq 1})$. ∎

**Remark 2.10** *A pure resolution $\mathbb{F}_{X,\bullet} = \{(F_q,\ \partial_q)\}_{q\geq 0}$ of type $(d_1, d_2, \ldots, d_\ell)$ is a grade $S$-free resolution of the form : $F_0 = S$, $F_q = S(-d_q)^{r_q}$ $(1 \leq q \leq \ell)$ and $d_1 < d_2 < \ldots < d_\ell$.*

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