Infinitesimal directions for strong Betti constancy in the Hilbert scheme of $\mathbb{P}^N(\mathbb{C})$

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Abstract

Assuming the arithmetic D_2-condition of a given closed subscheme $X \subseteq \mathbb{P}^N(\mathbb{C})$, we determine the infinitesimal directions at the closed point $[X]$ in the Hilbert scheme of $\mathbb{P}^N(\mathbb{C})$ for the constancy on the generalized graded Betti numbers holding.
Keywords: Hilbert scheme, infinitesimal direction, graded Betti number

§1 Introduction.

Let X be a closed subscheme of a projective space $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ whose Hilbert polynomial is $A(m) = A_X(m)$. Take the homogeneous coordinate ring $R_X = S/\mathbb{I}_X$ of X and its graded Betti numbers $\{b_{q,m}(X) = \dim_{\mathbb{C}}(Tor_q^S(R_X, S/S_+)_{(m)})\}_{q,m}$, where S_+ denotes the irrelevant maximal ideal of the polynomial ring $S = \mathbb{C}[Z_0, \ldots, Z_N]$ and $\mathbb{I}_X = \oplus_m H^0(P, I_X(m))$. Now we consider the universal family $\mathcal{U}$ of projective and flat embedded deformations of X in P, namely the family $\mathcal{U}$ parametrized by the Hilbert scheme $Hilb_P^{A(m)}$. We choose another closed subscheme Y of P which is obtained by a projective and flat embedded deformation of X over a connected algebraic scheme over $\mathbb{C}$. Since the Hilbert polynomial is an invariant by projective and flat embedded deformations, the Hilbert polynomial of X and that of Y are the same, namely $[Y] \in Hilb_P^{A(m)}$, but the equalities $b_{q,m}(X) = b_{q,m}(Y)$ $(\forall q,\ m \in \mathbb{Z})$ do not hold in general. Moreover, the graded Betti numbers does not satisfy in general even the upper semi-continuity which are satisfied by the cohomological invariants.

Now we suppose moreover that the subscheme X satisfies the arithmetic D_2-condition, namely the depth of R_X at the irrelevant maximal ideal S_+ is at least 2. It is easy to show that in a sufficiently small Zariski open neighborhood $V \subseteq Hilb_P^{A(m)}$ of the point $[X]$, any closed subscheme Y of P corresponding to a closed point $[Y] \in V$ also satisfies the arithmetic D_2-condition. If we restrict ourselves to the open set V, then the graded Betti numbers become cohomological invariants and satisfy the upper semi-continuity. Thus the "Betti constant set of X" $BC(X) := \{[Y] \in V \mid b_{q,m}(Y) = b_{q,m}(X) \quad (\forall q, m \in \mathbb{Z})\}$ is a locally closed set in V. However, by a technical reason, the set $BC(X)$ is still hard to study in general.

Hence, by using higher cohomologies, we introduce generalized graded Betti numbers $\{b_m^{p;q}\}$, which satisfy $b_{q,m} = b_m^{1;q}$ $(\forall q, m \in \mathbb{Z})$. Let us set $sBC(X) = \{[Y] \in V \mid b_m^{p;q}(Y) = b_m^{p;q}(X) \quad (\forall p, q, m \in \mathbb{Z})\}$ which will be called as the "strongly Betti constant set of X", whose nilpotent structure is given through a stratification with keeping the universality. By using the upper semi-continuity of cohomological invariants, it is easy to see that the set $sBC(X)$ is also a locally closed set. Obviously $sBC(X) \subseteq BC(X)$. In this article, we present a criterion for a given tangent vector σ of $Hilb_P^{A(m)}$ at the point $[X]$ (i.e.

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identified with a global normal vector field $\sigma \in H^0(N_X)$ of X in P by the theory of Hilbert schemes, cf. [2], [5], [7]) being included in the tangent space $T_{sBC(X),[X]}$. From the universality, this criterion is the same as a criterion for the given projective and flat family over the ring of dual numbers being strongly Betti constant.

In this article, if we are in the misleading cases of confusing global objects with sheaves, we attach underlines to the symbols of sheaves to emphasize being sheaves such as $\underline{Hom}$ in the classical way. On the theory of cohomological bease change, we often refer to the theorems in §12 of Chap. III in [4] for convenience, but we recommend also [3] Chap. III §7 and [6] Lect. 7 as excellent reference.

§2 Results.

Everything considered in this article is defined over the complex number field $\mathbb{C}$. For an arithmetic D_2-closed subscheme of a projective space, we generalize the definition of the ordinary graded Betti numbers as follows.

Definition 2.1 (generalized graded Betti numbers) *Let X be a closed subscheme of a projective space $P = \mathbb{P}^N(\mathbb{C})$ which satisfies the arithmetic D_2-condition. For integers $p, q, m \in \mathbb{Z}$, we define* the (generalized graded) Betti numbers *of X by*

$$b_m^{p,q} = b_m^{p,q}(X) = h^p(\Omega_P^q(m) \otimes I_X)$$

if $p \geq 0$, $0 \leq q \leq N$, $m \geq 1$, otherwise $b_m^{p,q} = 0$.

Remark 2.2 *The relation between our generalized graded Betti numbers $\{b_m^{p,q}\}$ and the (ordinary) graded Betti numbers $\{b_{q,m}\}$ is $b_{q,m} = b_m^{1,q}$. To confirm this equality, take the Koszul complex $\mathbb{K}_\bullet$ for the regular sequence $\{Z_0, \ldots, Z_N\}$ as the minimal graded S-free resolution of S/S_+ and compute $Tor_q^S(R_X, S/S_+)_{(m)}$, which is isomorphic to $H_q(R_X \otimes_S \mathbb{K}_\bullet)_{(m)} \cong H^1(\Omega_P^q(m) \otimes I_X)$ with using $R_X \cong \oplus_k H^0(X, O_X(k))$ and $\oplus_k H^1(P, I_X(k)) = 0$.*

There is a similar way of handling the group $Tor_q^S(R_X, S/S_+)_{(m)}$ to ours, which can be found in [1] etc.. However, there is a slight difference between ours and theirs even in the lowest level case: H^1. As an example, for an arithmetic D_2 closed subscheme $X \subseteq P = \mathbb{P}^N(\mathbb{C})$, the group $Tor_q^S(R_X, S/S_+)_{(m)}$ has a natural isomorphism to our group $H^1(\Omega_P^q(m) \otimes I_X)$, but has only a natural injection to their group $H^1(\Omega_P^{q+1}(m) \otimes O_X)$. Hence, to construct a universal family with keeping strictly the graded Betti numbers through a suitable stratification of the Hilbert scheme with respect to higher direct image sheaves, we can not use the higher direct image sheaves relating to the groups $\{H^p(\Omega_P^{q+1}(m) \otimes O_X)\}$.

Let us give a definition of the strongly Betti constant families.

Definition 2.3 (strongly Betti constant family) *Let B a connected algebraic scheme over $\mathbb{C}$, $P = \mathbb{P}^N(\mathbb{C})$, and $f = \pi|_{\mathfrak{X}} : \mathfrak{X} \to B$ a projective and flat morphism as in the commutative diagram (#-1), where $\mathfrak{X}$ and "incl." denote a closed subscheme of $P \times B$ and an inclusion morphism, respectively. Suppose that for any closed point $b \in B$, the fiber $X_b = f^{-1}(b)$ is an arithmetic D_2-closed subscheme of P (N.B. hence $R^1\pi_*(I_{\mathfrak{X}}(m)) = 0$ for any integer $m \in \mathbb{Z}$ including the case $m \leq 0$).*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{incl.} & P \times B \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \pi = pr_B} \\ & & B \end{array} \qquad (\#\text{-}1)$$

We say that the family $f : \mathfrak{X} \to B$ is strongly Betti constant *if the following condition* (†) *is satisfied.*

<u>*Condition* (†)</u>: *The coherent sheaves $R^p\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}})$ are O_B-locally free sheaves for all* (*non-negative*) *integers p, q, m with $m \geq 1$.*

Remark 2.4 *By applying Theorem 12.11 (b) in [4] and descending induction on p by starting from $p = N + 1$, it is easy to see that the condition* (†) *of Definition 2.3 is equivalent to the condition* (‡) *in the sequel.*

<u>*Condition* (‡)</u>: *For any closed point $b \in B$ and for all* (*non-negative*) *integers p, q, m with $m \geq 1$, the natural map $R^p\pi_*(\Omega^q_{P\times B/B}(m) \otimes I_{\mathfrak{X}}) \otimes k(b) \to H^p(\Omega^q_P(m) \otimes I_{X_b})$ is surjective.*

Then by applying Theorem 12.11 (a) in [4], we see that all the maps in condition (‡) *are isomorphisms, which implies that for all integers p, q, m with $m \geq 1$, the generalized graded Betti numbers $b^{p,q}_m$ of the closed fibers of the strongly Betti constant family $f : \mathfrak{X} \to B$ are the constants. If the base scheme B is reduced, the converse is also true (cf. Corollary 12.9* (*Grauert's Theorem*) *in [4]), namely the constancy of all the generalized graded Betti numbers implies that the family is strongly Betti constant.*

To explain the Key equality (#-8) in Theorem 2.6, let us confirm our notation and setting on a family over the ring of dual numbers.

We put $D_\varepsilon = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, namely the ring of dual numbers, $T_\varepsilon = Spec(D_\varepsilon)$, $t_0 \in T_\varepsilon$ the unique closed point. Let Y be a projective scheme over $\mathbb{C}$, X a closed subscheme of Y and $I_X \subseteq O_Y$ the sheaf of ideals defining X in Y. Take a global section σ of the normal sheaf $N_X = \underline{Hom}_{O_X}(I_X/I_X^2, O_X) \cong \underline{Hom}_{O_Y}(I_X, O_X)$ of X in Y, and consider the first infinitesimal embedded deformation $\mathfrak{X} = \mathfrak{X}_\sigma$ of X in Y corresponding to the section σ.

$$\begin{array}{ccccc} X & \xrightarrow{incl.} & \mathfrak{X} & & \\ {\scriptstyle incl.}\downarrow & & \downarrow{\scriptstyle incl.} & & \\ Y & \xrightarrow{incl.} & Y_\varepsilon = Y \times T_\varepsilon & \xrightarrow{h=pr_1} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v=pr_2} & & \downarrow{\scriptstyle u} \\ t_0 & \xrightarrow[incl.]{} & T_\varepsilon & \xrightarrow[s]{} & Spec(\mathbb{C}) \end{array} \qquad (\#\text{-}2)$$

Here, the symbols *incl.*, s, and u in the diagram (#-2) denote the inclusion morphism, the structure morphism of T_ε, and the one of Y, respectively. Take an r-bundle F on Y, namely an O_Y-locally free sheaf of rank r and set an O_{Y_ε}-locally free sheaf $E = h^*F$ to be the pull back of the bundle F by the morphism h. Let us consider the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_X & \xrightarrow{\times\varepsilon} & I_{\mathfrak{X}} & \longrightarrow & I_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_Y & \xrightarrow{\times\varepsilon} & O_{Y_\varepsilon} & \longrightarrow & O_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_X & \xrightarrow{\times\varepsilon} & O_{\mathfrak{X}} & \longrightarrow & O_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{\#-3}$$

Then $O_{Y_\varepsilon} = O_Y \cdot 1 \oplus O_Y \cdot \varepsilon$, and the structure sheaf $O_{\mathfrak{X}}$ of $\mathfrak{X}$ is the pushout for the pair of homomorphisms $I_X \hookrightarrow O_Y \cdot 1$ and $\sigma : I_X \to O_X \cdot \varepsilon$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_X & \xrightarrow{incl.} & O_Y & \longrightarrow & O_X & \longrightarrow & 0 \\
 & & \sigma\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_X & \xrightarrow[\times\varepsilon]{} & O_{\mathfrak{X}} & \longrightarrow & O_X & \longrightarrow & 0,
\end{array}$$

which has a natural ring structure by the principle of idealization. By the rough stalkwise expression, we see

$$O_{\mathfrak{X}} = (O_Y \cdot 1 \oplus O_X \cdot \varepsilon)/\{(z, -\sigma(z)\varepsilon) \mid z \in I_X\}.$$

The embedding $\mathfrak{X} \hookrightarrow Y_\varepsilon$, or the surjective homomorphism $O_{Y_\varepsilon} \to O_{\mathfrak{X}}$ is given stalkwisely by the composition of natural homomorphisms:

$$O_{Y_\varepsilon} = O_Y 1 \oplus O_Y \varepsilon \ \to \ O_Y 1 \oplus O_X \varepsilon \ \xrightarrow{can.} \ O_{\mathfrak{X}}.$$

Then the ideal sheaf $I_{\mathfrak{X}} \subseteq O_{Y_\varepsilon}$ which defines $\mathfrak{X}$ in Y_ε, is presented stalkwisely by

$$I_{\mathfrak{X}} = \{(a, b\varepsilon) \in O_Y 1 \oplus O_Y \varepsilon \mid a \in I_X,\ b \in O_Y,\ \text{s.t.}\quad \overline{b} = -\sigma(a)\ (\text{in } O_X)\ \},$$

where $\overline{b}$ denotes the image of a local section b of O_Y by the canonical homomorphism $O_Y \to O_X$.

As topological spaces, $|X| = |\mathfrak{X}|$, and $|Y| = |Y_\varepsilon|$ through the homeomorphism $|h|$ associated to the morphism h. Thus we always use the same symbol for a Zariski open set in $|Y_\varepsilon|$ as for the Zariski open set in $|Y|$ corresponding to the one through the homeomorphism $|h|$.

Let us take an affine open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of $|Y|$ which trivializes locally the bundle F and a local frame $\{f_{\alpha,i}\}_{i=1}^r$ on the affine open set U_α. Then we have

$$F|_{U_\alpha} = \bigoplus_{i=1}^{r} (O_Y|_{U_\alpha}) \cdot f_{\alpha,i} \tag{\#-4}$$

and on the affine open set $U_\alpha \cap U_\beta$, gluing data of F, i.e. a relation of the two local frames $\{f_{\alpha,i}\}_{i=1}^r$ and $\{f_{\beta,i}\}_{i=1}^r$:

$$(f_{\beta,1},\ldots,f_{\beta,r}) = (f_{\alpha,1},\ldots,f_{\alpha,r})\; G_{\alpha\beta}, \tag{\#-5}$$

where $G_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, GL(r, O_Y))$, namely $G_{\alpha\beta} = [g_{\alpha\beta,j,i}]$ is an $r \times r$-matrix with a (j,i)-element $g_{\alpha\beta,j,i} \in \Gamma(U_\alpha \cap U_\beta, O_Y)$ which satisfies $det(G_{\alpha\beta}) \in \Gamma(U_\alpha \cap U_\beta, O_Y^\times)$. The relation (#-5) can be written also as

$$f_{\beta,i} = \sum_{j=1}^{r} f_{\alpha,j}\; g_{\alpha\beta,j,i} \qquad (i = 1,\ldots,r). \tag{\#-6}$$

The local structure (#-4) induces the local structure of E on U_α as follows.

$$E|_{U_\alpha} = \bigoplus_{i=1}^{r} (O_Y|_{U_\alpha} 1 \oplus O_Y|_{U_\alpha} \varepsilon) \cdot f_{\alpha,i} \tag{\#-7}$$

The gluing data of E on the affine open set $U_\alpha \cap U_\beta$ is $G_{\alpha\beta} \cdot 1$, in other words, the relations (#-5) and (#-6) hold also for E, which means that through the gluing process, $G_{\alpha\beta} \cdot 1$ never mix the $O_Y \cdot 1$ part with the $O_Y \cdot \varepsilon$ part in $E|_{U_\alpha \cap U_\beta}$. This fact is an important key in our argument in the sequel.

Now let us make tensor products over O_{Y_ε} with the O_{Y_ε}-locally free sheaf E and the modules in the first row of the exact commutative diagram (#-3). Since I_X is the O_Y-module, we obtain

$$0 \longrightarrow F \otimes I_X \xrightarrow[\alpha_{IDF}]{\times\varepsilon} E \otimes I_{\mathfrak{X}} \xrightarrow[\beta_{IDF}]{} F \otimes I_X \longrightarrow 0,$$

which induces a long cohomology exact sequence and a connecting homomorphism $\delta^{(p)}_{IDF}$:

$$\cdots \longrightarrow H^p(E \otimes I_{\mathfrak{X}}) \xrightarrow{\beta_{IDF}} H^p(F \otimes I_X) \xrightarrow{\delta^{(p)}_{IDF}} H^{p+1}(F \otimes I_X) \xrightarrow{\alpha_{IDF}} \cdots .$$

Remark 2.5 *For the morphism $v : Y_\varepsilon \to T_\varepsilon$ and the unique closed point $t_0 \in T_\varepsilon$, the cohomological base change, or equivalently the surjectivity of the canonical map $R^p v_*(E \otimes I_{\mathfrak{X}}) \otimes k(t_0) \to H^p(F \otimes I_X)$ is equivalent to $\delta^{(p)}_{IDF} = 0$.*

Since the third column in the exact commutative diagram (#-3) is a short exact sequence of O_Y-modules, we can make tensor products over O_Y with the O_Y-locally free sheaf F and the modules in the third column. Then we have

$$0 \longrightarrow F \otimes I_X \xrightarrow{\alpha_{LFT}} F \xrightarrow{\beta_{LFT}} F \otimes O_X \longrightarrow 0,$$

which induces a long cohomology exact sequence and a connecting homomorphism $\delta^{(p)}_{LFT}$:

$$\cdots \longrightarrow H^p(F) \xrightarrow{\beta_{LFT}} H^p(F \otimes O_X) \xrightarrow{\delta^{(p)}_{LFT}} H^{p+1}(F \otimes I_X) \xrightarrow{\alpha_{LFT}} \cdots .$$

The (global) normal vector field σ along X, namely the global section $\sigma \in \Gamma(X, N_X)$ can be considered as an element $\sigma \in Hom_{O_Y}(I_X, O_X)$ and induces naturally a homomorphism $\sigma : H^p(F \otimes I_X) \to H^p(F \otimes O_X)$.

Theorem 2.6 *Under the circumstances, we have an equality of the maps:*

$$\delta^{(p)}_{IDF} = -\delta^{(p)}_{LFT} \circ \sigma. \tag{\#-8}$$

Proof. Take the affine open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of the space $|Y_\varepsilon| = |Y|$ and the (common) local frames $\{f_{\alpha,i}\}_{i=1}^r$ ($\alpha \in A$) of F and of E with respect to the covering $\mathfrak{U}$ as above. To check the equality (#-8), after tensoring E to the exact commutative diagram (#-3), we use the Čech complexes $\{\mathcal{C}^\bullet(\mathfrak{U}, -), \check{\delta}\}$ of them with respect to the covering $\mathfrak{U}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F \otimes I_X) & \xrightarrow{\times\varepsilon} & \mathcal{C}^\bullet(\mathfrak{U}, E \otimes I_{\mathfrak{X}}) & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F \otimes I_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F) & \xrightarrow{\times\varepsilon} & \mathcal{C}^\bullet(\mathfrak{U}, E) & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F \otimes O_X) & \xrightarrow{\times\varepsilon} & \mathcal{C}^\bullet(\mathfrak{U}, E \otimes O_{\mathfrak{X}}) & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{U}, F \otimes O_X) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{\#-9}$$

As usual, we denote $U_{\alpha_0,\alpha_1,\ldots,\alpha_p}$ for the open set $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$. On the open set $U_{\alpha_0,\alpha_1,\ldots,\alpha_p}$, to trivialize the O_{Y_ε}-locally free sheaf E or the O_Y-locally free sheaf F, we almost always take the local frame $\{f_{\alpha_0,i}\}_{i=1}^r$ on the open set U_{α_0} whose index is the first index of the open set $U_{\alpha_0,\alpha_1,\ldots,\alpha_p}$, otherwise mentioned particularly.

To see the gluing data of E more precisely, we take a local section $\gamma \in \Gamma(V, E)$, where the open set V is included in $U_{\alpha,\beta}$. Then

$$\gamma = \sum_{j=1}^r (\lambda_{\alpha,j} + \varepsilon\mu_{\alpha,j}) f_{\alpha,j} = \sum_{i=1}^r (\lambda_{\beta,i} + \varepsilon\mu_{\beta,i}) f_{\beta,i}$$

where $\lambda_{\alpha,j}, \mu_{\alpha,j}, \lambda_{\beta,i}, \mu_{\beta,i} \in \Gamma(V, O_Y)$. By using the gluing data (#-6) for E, the fiber coordinate transformation of E, namely the relation of the coefficient functions in γ is written as follows.

$$\lambda_{\alpha,j} = \sum_{i=1}^r g_{\alpha\beta,j,i} \lambda_{\beta,i} \qquad \mu_{\alpha,j} = \sum_{i=1}^r g_{\alpha\beta,j,i} \mu_{\beta,i} \tag{\#-10}$$

To show the equality (#-8), by taking any element $\overline{\omega} \in H^p(F \otimes I_X)$, it is enough to show that $\delta^{(p)}_{IDF}(\overline{\omega}) = -\delta^{(p)}_{LFT} \circ \sigma(\overline{\omega})$ in the space $H^{p+1}(F \otimes I_X)$. Let us take a Čech representative $\omega \in \mathcal{C}^p(\mathfrak{U}, F \otimes I_X)$ of the element $\overline{\omega} \in H^p(F \otimes I_X)$ with $\omega = \{(U_{\alpha_0,\alpha_1,\ldots,\alpha_p},\ \omega_{\alpha_0,\alpha_1,\ldots,\alpha_p})\}$ and

$$\omega_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \sum_{j=1}^r \varphi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\ j} \cdot f_{\alpha_0,\ j} \tag{\#-11}$$

where $\varphi_{\alpha_0,\alpha_1,\dots,\alpha_p,\, j} \in \Gamma(U_{\alpha_0,\alpha_1,\dots,\alpha_p}, I_X)$. The cocycle condition of $\omega \in \mathcal{C}^p(\mathfrak{U}, F \otimes I_X)$ is $\overset{\vee}{\delta}\,(\omega) = \{(U_{\beta_0,\beta_1,\dots,\beta_{p+1}},\ (\overset{\vee}{\delta}\,\omega)_{\beta_0,\beta_1,\dots,\beta_{p+1}})\} = 0$ in $\mathcal{C}^{p+1}(\mathfrak{U}, F \otimes I_X)$, namely,

$$0 = (\overset{\vee}{\delta}\,\omega)_{\beta_0,\beta_1,\dots,\beta_{p+1}} = \sum_{t=0}^{p+1}(-1)^t \left(\omega_{\beta_0,\dots\overset{t}{\smile}\dots,\beta_{p+1}}|_{U_{\beta_0,\dots,\beta_{p+1}}}\right),$$

which implies that, by omitting from now on the restriction symbol "$|_{U_{\beta_0,\cdots,\beta_{p+1}}}$" for simplicity,

$$\begin{aligned}
0 &= \omega_{\beta_1,\dots,\beta_{p+1}} + \sum_{t=1}^{p+1}(-1)^t \omega_{\beta_0,\dots\overset{t}{\smile}\dots,\beta_{p+1}} && (\#\text{-}12)\\
&= \sum_{i=1}^{r} \varphi_{\beta_1,\dots,\beta_{p+1},\, i}\cdot f_{\beta_1,\, i} \ + \sum_{t=1}^{p+1}(-1)^t \left(\sum_{j=1}^{r} \varphi_{\beta_0,\dots\overset{t}{\smile}\dots,\beta_{p+1},\, j}\cdot f_{\beta_0,\, j}\right) && (\#\text{-}13)\\
&= \sum_{j=1}^{r}\left(\sum_{i=1}^{r} g_{\beta_0,\beta_1,\, j,i}\cdot \varphi_{\beta_1,\dots,\beta_{p+1},\, i} \ + \sum_{t=1}^{p+1}(-1)^t \varphi_{\beta_0,\dots\overset{t}{\smile}\dots,\beta_{p+1},\, j}\right)\cdot f_{\beta_0,\, j}. && (\#\text{-}14)
\end{aligned}$$

Here we make a remark that the expression (#-13) is useful in the following argument, which is an exceptional case to our previous engagement of taking local frames.

Thus, in terms of the coefficient functions, on the open set $U_{\beta_0,\dots,\beta_{p+1}}$, the cocycle condition is

$$\left(\sum_{i=1}^{r} g_{\beta_0,\beta_1,\, j,i}\cdot \varphi_{\beta_1,\dots,\beta_{p+1},\, i}\right) + \sum_{t=1}^{p+1}(-1)^t \varphi_{\beta_0,\dots\overset{t}{\smile}\dots,\beta_{p+1},\, j} = 0 \qquad (\#\text{-}15)$$

for $j = 1, \dots r$.

- The calculation of $\delta^{(p)}_{IDF}(\overline{\omega})$.

To study the connecting homomorphism $\delta^{(p)}_{IDF}$, let us recall a familiar exact commutative diagram which arises from the first row of the exact commutative diagram (#-9).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^p(\mathfrak{U}, F\otimes I_X) & \xrightarrow[\times\varepsilon]{\alpha_{IDF}} & \mathcal{C}^p(\mathfrak{U}, E\otimes I_{\mathfrak{X}}) & \xrightarrow{\beta_{IDF}} & \mathcal{C}^p(\mathfrak{U}, F\otimes I_X) & \longrightarrow & 0\\
 & & \big\downarrow \overset{\vee}{\delta}{}' & & \big\downarrow \overset{\vee}{\delta} & & \big\downarrow \overset{\vee}{\delta}{}'' & & \\
0 & \longrightarrow & \mathcal{C}^{p+1}(\mathfrak{U}, F\otimes I_X) & \xrightarrow[\alpha_{IDF}]{\times\varepsilon} & \mathcal{C}^{p+1}(\mathfrak{U}, E\otimes I_{\mathfrak{X}}) & \xrightarrow[\beta_{IDF}]{} & \mathcal{C}^{p+1}(\mathfrak{U}, F\otimes I_X) & \longrightarrow & 0
\end{array} \qquad (\#\text{-}16)$$

We take a lift $\widetilde{\omega} = \{(U_{\alpha_0,\alpha_1,\dots,\alpha_p},\ \widetilde{\omega}_{\alpha_0,\alpha_1,\dots,\alpha_p})\} \in \mathcal{C}^p(\mathfrak{U}, E\otimes I_{\mathfrak{X}})$ of the Čech representative $\omega \in \mathcal{C}^p(\mathfrak{U}, F\otimes I_X)$ of the element $\overline{\omega} \in H^p(Y, F\otimes I_X)$, namely $\omega = \beta_{IDF}(\widetilde{\omega})$. By tensoring E to the diagram (#-3), we see that the homomorphism β_{IDF} is compatible with the natural homomorphism given locally :

$$E|_{U_\alpha} = \bigoplus_{i=1}^{r}(O_Y|_{U_\alpha}1 \oplus O_Y|_{U_\alpha}\varepsilon)\cdot f_{\alpha,i} \ \longrightarrow\ F|_{U_\alpha} = \bigoplus_{i=1}^{r}(O_Y|_{U_\alpha}1)\cdot f_{\alpha,i},$$

and that the local section $\widetilde{\omega}_{\alpha_0,\alpha_1,\dots,\alpha_p} \in \Gamma(U_{\alpha_0,\alpha_1,\dots,\alpha_p}, E\otimes I_{\mathfrak{X}})$ has an expression :

$$\widetilde{\omega}_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \sum_{j=1}^{r} \left(1 \cdot \varphi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j} \;+\; \varepsilon \cdot \psi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j}\right) \cdot f_{\alpha_0,\, j}, \tag{\#-17}$$

where $\psi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j} \in \Gamma(U_{\alpha_0,\alpha_1,\ldots,\alpha_p}, O_Y)$ such that

$$\overline{\psi}_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j} = -\sigma(\varphi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j}) \tag{\#-18}$$

through the canonical homomorphism $O_Y \to O_X$ by $\lambda \mapsto \overline{\lambda}$ for a local section λ of O_Y. Then, by applying the Čech differential to the element $\widetilde{\omega}$ and imitating the process of getting (#-13), we have $\overset{\vee}{\delta}\,(\widetilde{\omega}) \in \mathcal{C}^{p+1}(\mathfrak{U}, E \otimes I_{\mathfrak{X}})$ with the form $\overset{\vee}{\delta}\,(\widetilde{\omega}) = \{(U_{\beta_0,\beta_1,\ldots,\beta_{p+1}},\ (\overset{\vee}{\delta}\,\widetilde{\omega})_{\beta_0,\beta_1,\ldots,\beta_{p+1}})\}$ which satisfies by omitting the restriction symbol "$|_{U_{\beta_0,\beta_1,\ldots,\beta_{p+1}}}$",

$$\begin{aligned}(\overset{\vee}{\delta}\,\widetilde{\omega})_{\beta_0,\beta_1,\ldots,\beta_{p+1}} \;=\;& \sum_{i=1}^{r} \left(\varphi_{\beta_1,\ldots,\beta_{p+1},\, i} \;+\; \varepsilon \cdot \psi_{\beta_1,\ldots,\beta_{p+1},\, i}\right) \cdot f_{\beta_1,\, i} \;+ \\ & \sum_{t=1}^{p+1} (-1)^t \left\{ \sum_{j=1}^{r} \left(\varphi_{\beta_0,\ldots\overset{t}{\smile}\ldots,\beta_{p+1},\, j} \;+\; \varepsilon \cdot \psi_{\beta_0,\ldots\overset{t}{\smile}\ldots,\beta_{p+1},\, j}\right) \cdot f_{\beta_0,\, j} \right\}.\end{aligned} \tag{\#-19}$$

Since the expression (#-13) is zero, we have

$$= \left\{ \sum_{i=1}^{r} \psi_{\beta_1,\ldots,\beta_{p+1},\, i} \cdot f_{\beta_1,\, i} \;+\; \sum_{t=1}^{p+1} (-1)^t \left(\sum_{j=1}^{r} \psi_{\beta_0,\ldots\overset{t}{\smile}\ldots,\beta_{p+1},\, j} \cdot f_{\beta_0,\, j} \right) \right\} \varepsilon. \tag{\#-20}$$

Now we take an element $\rho = \{(U_{\beta_0,\beta_1,\ldots,\beta_{p+1}},\ \rho_{\beta_0,\beta_1,\ldots,\beta_{p+1}})\} \in \mathcal{C}^{p+1}(\mathfrak{U}, F)$ with

$$\rho_{\beta_0,\beta_1,\ldots,\beta_{p+1}} = \sum_{i=1}^{r} \psi_{\beta_1,\ldots,\beta_{p+1},\, i} \cdot f_{\beta_1,\, i} \;+\; \sum_{t=1}^{p+1} (-1)^t \left(\sum_{j=1}^{r} \psi_{\beta_0,\ldots\overset{t}{\smile}\ldots,\beta_{p+1},\, j} \cdot f_{\beta_0,\, j} \right). \tag{\#-21}$$

By considering $\mathcal{C}^{p+1}(\mathfrak{U}, E \otimes I_{\mathfrak{X}}) \subseteq \mathcal{C}^{p+1}(\mathfrak{U}, E)$, the equality (#-20) shows that $\overset{\vee}{\delta}\,(\widetilde{\omega}) = \varepsilon \cdot \rho$ in $\mathcal{C}^{p+1}(\mathfrak{U}, E)$. This equality and diagram chasing on the exact commutative diagram of complexes (#-9) bring us that $\rho \in \mathcal{C}^{p+1}(\mathfrak{U}, F \otimes I_X)$ and $\overset{\vee}{\delta}\,(\rho) = 0$. These can also be confirmed directly on the module $\mathcal{C}^{p+1}(\mathfrak{U}, F \otimes O_X)$ by using the equalities (#-18) and (#-15) (or (#-13)). Thus we have $\delta^{(p)}_{IDF}(\overline{\omega}) = [\rho]$ in $H^{p+1}(Y, F \otimes I_X)$, where $[\rho]$ denotes the cohomology class defined by the cocycle ρ.

- The calculation of $\delta^{(p)}_{LFT} \circ \sigma(\overline{\omega})$.

Now we use again the Čech representative $\omega \in \mathcal{C}^p(\mathfrak{U}, F \otimes I_X)$ of the element $\overline{\omega}$. Then $\sigma(\overline{\omega}) = [\sigma(\omega)] \in H^p(Y, F \otimes O_X)$ and $\sigma(\omega) = \{(U_{\alpha_0,\alpha_1,\ldots,\alpha_p},\ \sigma(\omega)_{\alpha_0,\alpha_1,\ldots,\alpha_p})\} \in \mathcal{C}^p(\mathfrak{U}, F \otimes O_X)$ with

$$\sigma(\omega)_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \sum_{j=1}^{r} \sigma(\varphi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\, j}) \cdot f_{\alpha_0,\, j}. \tag{\#-22}$$

For computing the connecting homomorphism $\delta^{(p)}_{LFT}$, let us recall the following exact commutative diagram which is brought by the third column of the exact commutative diagram (#-9).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}^p(\mathfrak{U}, F \otimes I_X) & \xrightarrow{\alpha_{LFT}} & \mathcal{C}^p(\mathfrak{U}, F) & \xrightarrow{\beta_{LFT}} & \mathcal{C}^p(\mathfrak{U}, F \otimes O_X) & \longrightarrow & 0 \\
 & & \Big\downarrow \overset{\vee}{\delta'} & & \Big\downarrow \overset{\vee}{\delta}_F & & \Big\downarrow \overset{\vee}{\delta}_{F,X} & & \\
0 & \longrightarrow & \mathcal{C}^{p+1}(\mathfrak{U}, F \otimes I_X) & \xrightarrow[\alpha_{LFT}]{} & \mathcal{C}^{p+1}(\mathfrak{U}, F) & \xrightarrow[\beta_{LFT}]{} & \mathcal{C}^{p+1}(\mathfrak{U}, F \otimes O_X) & \longrightarrow & 0
\end{array}
\qquad (\#\text{-}23)
$$

Let us construct a lift $\widetilde{\sigma(\omega)} = \{(U_{\alpha_0,\alpha_1,\ldots,\alpha_p},\ \widetilde{\sigma(\omega)}_{\alpha_0,\alpha_1,\ldots,\alpha_p})\} \in \mathcal{C}^p(\mathfrak{U}, F)$ of the cocycle $\sigma(\omega)$ with respect to the homomorphism β_{LFT} by using the equality (#-18). Namely, we set

$$
\widetilde{\sigma(\omega)}_{\alpha_0,\alpha_1,\ldots,\alpha_p} = -\sum_{j=1}^{r} \psi_{\alpha_0,\alpha_1,\ldots,\alpha_p,\ j} \cdot f_{\alpha_0,\ j}. \qquad (\#\text{-}24)
$$

Then we apply Čech differential $\overset{\vee}{\delta}_F$ to the element $\widetilde{\sigma(\omega)}$ and get $\overset{\vee}{\delta}_F\ (\widetilde{\sigma(\omega)}) = \{(U_{\beta_0,\beta_1,\ldots,\beta_{p+1}}, \overset{\vee}{\delta}_F\ (\widetilde{\sigma(\omega)})_{\beta_0,\beta_1,\ldots,\beta_{p+1}})\} \in \mathcal{C}^{p+1}(\mathfrak{U}, F)$ which satisfies that by omitting the restriction symbol "$|_{U_{\beta_0,\ldots,\beta_{p+1}}}$" again,

$$
\overset{\vee}{\delta}_F\ (\widetilde{\sigma(\omega)})_{\beta_0,\beta_1,\ldots,\beta_{p+1}} = \widetilde{\sigma(\omega)}_{\beta_1,\ldots,\beta_{p+1}} + \sum_{t=1}^{p+1} (-1)^t\ \widetilde{\sigma(\omega)}_{\beta_0,\ldots\overset{t}{\smile}\ldots,\beta_{p+1}}. \qquad (\#\text{-}25)
$$

Here we apply our setting (#-24) to each term in (#-25) above and obtain from the equality (#-21) that

$$
\begin{aligned}
&= -\left\{ \sum_{i=1}^{r} \psi_{\beta_1,\beta_2,\ldots,\beta_{p+1},i} \cdot f_{\beta_1,\ i} + \sum_{t=1}^{p+1} (-1)^t \left(\sum_{j=1}^{r} \psi_{\beta_0,\ldots,\overset{t}{\vee}\ldots,\beta_{p+1},\ j} \cdot f_{\beta_0,j} \right) \right\} && (\#\text{-}26) \\
&= -\rho_{\beta_0,\beta_1,\ldots,\beta_{p+1}}. && (\#\text{-}27)
\end{aligned}
$$

Thus we have shown that $\delta^{(p)}_{IDF}(\overline{\omega}) = [\rho] = -\delta^{(p)}_{LFT} \circ \sigma(\overline{\omega})$, namely the equality (#-8).

∎

In the case that $Y = \mathbb{P}^N(\mathbb{C})$ and $F = \Omega^q_P(m)$, the connecting homomorphism $\delta^{(p,q)}_{LFT,m} := \delta^{(p)}_{LFT}$ in the equality (#-8) has the following property.

Lemma 2.7 *For the projective space $P = \mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ and a closed subscheme $X \subseteq P$, the natural exact sequence:*

$$
0 \to \Omega^q_P(m) \otimes I_X \to \Omega^q_P(m) \to \Omega^q_P(m) \otimes O_X \to 0 \qquad (\#\text{-}28)
$$

induces a lifting obstruction map with respect to X (cf. [8]):

$$
\delta^{(p,q)}_{LFT,m} : H^p(\Omega^q_P(m) \otimes O_X) \to H^{p+1}(\Omega^q_P(m) \otimes I_X)
$$

as a connecting homomorphism. Then this map $\delta^{(p,q)}_{LFT,m}$ is injective for integers p, q and m with $p \geq 1$, $0 \leq q \leq N$, and $m \geq 1$.

Proof. By taking a long exact sequence of (#-28), we have

$$H^p(\Omega_P^q(m)) \longrightarrow H^p(\Omega_P^q(m)\otimes O_X) \xrightarrow{\delta^{(p,q)}_{LFT,m}} H^{p+1}(\Omega_P^q(m)\otimes I_X),$$

apply the Bott vanishing theorem to $H^p(\Omega_P^q(m))$, and get the result directly. ∎

Remark 2.8 *For $p \geq N$, obviously $\delta^{(p,q)}_{LFT,m} = 0$ and $H^p(\Omega_P^q(m)\otimes O_X) = 0$.*

Applying Theorem 2.6 and Lemma 2.7 to the case that $Y = \mathbb{P}^N(\mathbb{C})$ and $F = \Omega_P^q(m)$, now we can determine the infinitesimal directions for strong Betti constancy holding at a closed point $[X]$ in the Hilbert scheme of $\mathbb{P}^N(\mathbb{C})$ which corresponds to an arithmetic D_2-closed subscheme $X \subseteq \mathbb{P}^N(\mathbb{C})$.

Main Theorem 2.9 *Let X be an arithmetic D_2-closed subscheme of a projective space $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ whose Hilbert polynomial is $A(m) = A_X(m)$. Take the Hilbert scheme $H = Hilb_P^{A(m)}$ which parametrizes all the closed subschemes of P whose Hilbert polynomials coincide with $A(m)$, and the tangent space $T_{H,[X]}$ of H at the point $[X]$, which is naturally isomorphic to the space of global normal vector fields $H^0(N_{X/P})$ of X in P. A section $\sigma \in H^0(N_{X/P}) \cong Hom_{O_P}(I_X, O_X)$ corresponds to a projective and flat family $f : \mathfrak{X} = \mathfrak{X}_\sigma \to T_\varepsilon = Spec(D_\varepsilon)$ with $\mathfrak{X} \subseteq P \times T_\varepsilon$, $f = pr_{T_\varepsilon}|_{\mathfrak{X}}$, and $f^{-1}(t_0) = X$ for the unique closed point $t_0 \in T_\varepsilon$.*

Then the family $f : \mathfrak{X} \to T_\varepsilon$ is strongly Betti constant if and only if the followoing condition (*) holds.

Condition(*)

The natural induced map $\sigma = \sigma_m^{p,q} : H^p(\Omega_P^q(m)\otimes I_X) \to H^p(\Omega_P^q(m)\otimes O_X)$ by $\sigma \in H^0(N_{X/P})$ is a zero map ($\forall p,q,m \in \mathbb{Z}$ with $p,m \geq 1$).

In other words,

$$\begin{aligned} T_{sBC(X),[X]} &= \{\sigma \in H^0(N_{X/P}) \cong T_{H,[X]} \mid \sigma_m^{p,q} = 0 \textit{ as the map} \\ &: H^p(\Omega_P^q(m)\otimes I_X) \to H^p(\Omega_P^q(m)\otimes O_X) \quad (\forall p,q,m \in \mathbb{Z}, \textit{ with } p,m \geq 1)\}. \end{aligned}$$

Proof. The point of poof is to handle the difference between the condition (†) in Definition 2.3 which includes the case $p = 0$ and the condition (*) above which does not include the case $p = 0$.

First we consider the trivial diagram:

$$\begin{array}{ccc} P_\varepsilon = P \times T_\varepsilon & \xrightarrow{h} & P \\ v\downarrow & & \downarrow u \\ T_\varepsilon & \xrightarrow[s]{} & Spec\mathbb{C}. \end{array} \qquad (\#\text{-}29)$$

It is well-known that $\Omega^q_{P_\varepsilon/T_\varepsilon}(m) \cong h^*\Omega^q_P(m)$.

To simplify the proof, for fixed integers p,q,m, we introduce the following two abbreviations on the conditions of locally freeness and of the cohomological base change.

$(L.F.)_m^{p,q}$: $R^p v_*(\Omega^q_{P_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})$ is an O_{T_ε}-locally free sheaf.

$(C.B.C.)^{p,q}_m$: The natural map $R^p v_*(\Omega^q_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \otimes k(t_0) \to H^p(\Omega^q_P(m) \otimes I_X)$ is surjective.

Now we assume strong Betti constancy. By Remark 2.4, the strong Betti constancy is equivalent to both of the two conditions (†) and (‡). Hence, the natural map $R^p v_*(\Omega^q_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \cong H^p(\Omega^q_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \to H^p(\Omega^q_P(m) \otimes I_X)$ is surjective for any non-negative integer p, q, and m with $m \geq 1$. In other words, by putting $Y = P$, and $F = \Omega^q_P(m)$, the connecting homomorphism $\delta^{(p)}_{IDF} : H^p(\Omega^q_P(m) \otimes I_X) \to H^{p+1}(\Omega^q_P(m) \otimes I_X)$ is a zero map for any non-negative integer p, q, and m with $m \geq 1$. Then we apply Theorem 2.6 and Lemma 2.7 and easily obtain the condition (*).

Conversely, we assume the condition (*). Apply Theorem 2.6 by putting $Y = P$, and $F = \Omega^q_P(m)$, we see that the connecting homomorphism $\delta^{(p)}_{IDF} : H^p(\Omega^q_P(m) \otimes I_X) \to H^{p+1}(\Omega^q_P(m) \otimes I_X)$ is a zero map for any non-negative integer p, q, and m with $p,\ m \geq 1$. Namely we get the condition $(C.B.C.)^{p,q}_m$ for all non-negative integers p, q, m with $p, m \geq 1$. Then, by applying Theorem 12.11 (b) in [4] and descending induction on p by starting from $p = N + 1$, we obtain the condition $(L.F.)^{p,q}_m$ for all integers p, q, m with $p \geq 2$ and $m \geq 1$. It is left to show the condition $(C.B.C.)^{0,q}_m$ for all non-negative integers q, m with $m \geq 1$.

Since we assume the arithmetic D_2-condition on X, we have $H^1(\Omega^N_P(m) \otimes I_X) \cong H^1(I_X(m-N-1)) = 0$, which and Nakayama's lemma bring $R^1 v_*(\Omega^N_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) = R^1 v_*(I_{\mathfrak{X}}(m-N-1)) = 0$ through Theorem 12.11 (a) in [4]. Namely we have $(C.B.C.)^{1,N}_m$ and $(L.F.)^{1,N}_m$ for any integer $m \geq 1$. Then, Theorem 12.11 (b) in [4] shows $(C.B.C.)^{0,N}_m$ for any integer $m \geq 1$. The condition $(C.B.C.)^{-1,N}_m$ is trivial. Hence, Theorem 12.11 (b) in [4] shows $(L.F.)^{0,N}_m$. Thus we show that in the case : $q = N$, all the conditions $(C.B.C.)^{p,N}_m$ and $(L.F.)^{p,N}_m$ hold for any integers $p \geq 0$ and $m \geq 1$.

It should be noticed that since $\Omega^N_{P_\varepsilon/T_\varepsilon}(m) \cong O_{P_\varepsilon}(m-N-1)$, the conditions $(C.B.C.)^{0,N}_m$ and $(L.F.)^{0,N}_m$ imply that the direct image sheaf $v_*(I_{\mathfrak{X}}(m'))$ is of O_{T_ε}-locally free and the cohomological base change $v_*(I_{\mathfrak{X}}(m')) \otimes k(t_0) \cong H^0(I_X(m'))$ holds for any integer $m' \geq -N$.

Starting from $q = N$, we apply the descending induction on q to show all the conditions $(C.B.C.)^{p,q}_m$ and $(L.F.)^{p,q}_m$ for any integers $p \geq 0$ and $m \geq 1$. The induction step from q to $q - 1$ goes as follows.

By our induction hypothesis, we may assume that $1 \leq q \leq N$, the conditions $(C.B.C.)^{p,q}_m$ and $(L.F.)^{p,q}_m$ hold for any integers $p \geq 0$ and $m \geq 1$. Moreover, we have already proved that the conditions $(C.B.C.)^{p,q-1}_m$ for integers $p \geq 1$ and $m \geq 1$ and the conditions $(L.F.)^{p,q-1}_m$ hold for integers $p \geq 2$ and $m \geq 1$.

We recall a short exact sequences :

$$0 \longrightarrow \Omega^q_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}} \longrightarrow \oplus I_{\mathfrak{X}}(m-q) \longrightarrow \Omega^{q-1}_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}} \longrightarrow 0, \qquad (\#\text{-}30)$$

apply v_* and obtain an exact sequence :

$$\oplus v_*(I_{\mathfrak{X}}(m-q)) \longrightarrow v_*(\Omega^{q-1}_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \longrightarrow R^1 v_*(\Omega^q_{P_\varepsilon/T_\varepsilon}(m) \otimes I_{\mathfrak{X}}) \longrightarrow 0. \qquad (\#\text{-}31)$$

Tensoring $k(t_0)$ to the sequence (#-31) and compare it with the long cohomology exact sequence of the next sequence :

$$0 \longrightarrow \Omega^q_P(m) \otimes I_X \longrightarrow \oplus I_X(m-q) \longrightarrow \Omega^{q-1}_P(m) \otimes I_X \longrightarrow 0 \qquad (\#\text{-}32)$$

from the view point of the cohomological base change. Then we have an exact commutative diagram:

$$\begin{array}{ccccccc}
\oplus v_*(I_{\mathfrak{X}}(m-q))\otimes k(t_0) & \longrightarrow & v_*(\Omega^{q-1}_{P_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})\otimes k(t_0) & \longrightarrow & R^1v_*(\Omega^{q}_{P_\varepsilon/T_\varepsilon}(m)\otimes I_{\mathfrak{X}})\otimes k(t_0) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow & & \downarrow\cong & & \\
\oplus H^0(I_X(m-q)) & \longrightarrow & H^0(\Omega^{q-1}_P(m)\otimes I_X) & \longrightarrow & H^1(\Omega^q_P(m)\otimes I_X) & \longrightarrow & 0.
\end{array}$$

(#-33)

An easy diagram chasing in (#-33) shows the condition $(C.B.C.)^{0,q-1}_m$ for any integer $m \geq 1$. Since we already have the condition $(C.B.C.)^{1,q-1}_m$, we get the condition $(L.F.)^{1,q-1}_m$. As a result, we have $(C.B.C.)^{p,q-1}_m$ for all integers p,m with $p \geq 0$ and $m \geq 1$, which implies $(L.F.)^{p,q-1}_m$ for all integers p,m with $p,m \geq 1$. On the other hand $(C.B.C.)^{-1,q-1}_m$ is trivial, which brings $(L.F.)^{0,q-1}_m$.

Thus we see the strong Betti constancy of the family $f : \mathfrak{X} \to T_\varepsilon$ holding. ∎

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