Good homological shells of a smooth projective curve given by an intersection of two Segre 3-folds in $\mathbb{P}^5(\mathbb{C})$

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Abstract

Inspired by our recent results [16] and [17], we generalize one of our previous problems into "Parallelogram Conjecture" (cf. Working Hypothesis 1.15). From the view point of this conjecture, we study good homological shells of a projective curve $X \subseteq \mathbb{P}^5(\mathbb{C})$ which is presented as a transverse intersection of two Segre 3-folds V and V^σ in general position. In particular, we show that V and V^σ are isolated in the universal family of homological shells of X. It is already known that the number of the weak shell equivalence classes of a fixed projective subvariety is always finite (cf. [18]). And this example shows that for a fixed Hilbert polynomial, there may exist more than one weak shell equivalence classes. From this example, we can also see that even for a fixed Hilbert polynomial, the parameter space of the universal family of homological shells with the Hilbert polynomial may not be connected.
Keywords: Segre 3-fold, cut by an arthmetically Cohen-Macaulay scheme, homological shell (=pre-geometric shell), good homological shell

§0 Introduction.

Also in this article, it is still our main concern to study the "geometric structure" of a given projective embedding of a projective variety from the view point of homological shells (cf.[8]∼[18]). In our paper [15], we raised a problem on the structure of the graph of homological shells of a projective subvariety $X \subseteq \mathbb{P}^N = P$ which is presented by a hypersurface cut of a projective variety W with $\dim(W) = \dim(X) + 1$ (cf. Problem 1.14 below). After the paper [15], we introduced newly two kinds of equivalence relations on homological shells (cf. [16]). Hence we can consider new types of graphs, the graphs of homological shells up to these equivalence relations. On the other hand, in [17], we generalized the hypersurface cut to the cut by a locally Cohen-Macaulay projective subscheme with relatively large arithmetic depth.

Then, these new points of view bring us a generalization of that problem as a new working hypothesis (cf. Working Hypothesis 1.15), which will be called as "Parallelogram Conjecture".

To verify this working hypothesis by studying homological shells of a concrete example, we need a test example with a relatively small codimension which is not obtained by successive hypersurface cuts of a projective variety. Thus we consider good homological shells of a smooth projective curve $X \subseteq \mathbb{P}^5(\mathbb{C})$ of degree 9 which is a transverse intersection of two Segre 3-folds one of which is translated generally by $PGL(6, \mathbb{C})$-action.

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We should make a remark that to handle this example, it is enough to apply Prof.M.Hashimoto's Remark (cf. [8] Proposition 1.4) and not necessary to apply the main result of [17]. It would be interesting to study examples which need truly the result of [17] (cf. Theorem 1.6).

The author would like to express his deep gratitude to Prof. A. Ohbuchi for answering to his question on the curves with satisfying the condition described above by constructing a good example of a curve embedded by a super canonical linear system. It was mysterious that this curve is slightly different from the curve which the author independently constructed by using another super canonical linear system. At this stage, this mystery is not really illuminated yet but which gives a stimulation to study the objects handled in this paper.

§1 Preliminaries.

Let us recall our two fundamental conjectures on homological shells from [8] and [9]. By the influence of [15], we refine the claims of these conjectures.

Conjecture 1.1 *Let $P = \mathbb{P}^N(\mathbb{C})$ be an N-th projective space with the tautological ample line bundle $O_P(1) = O_P(H)$ and $V \subseteq W \subseteq P$ its closed subschemes.*

(1.1.1) *Assume that the scheme V is a variety, namely reduced and irreducible and that the closed subscheme W is a (good) homological shell of V. Then the subscheme W is also a variety (if necessary, we may move W up to weak shell equivalence).*

(1.1.2) [**Δ-genus inequality conjecture**] *Suppose that the subscheme V is arithmetically D_2, namely its arithmetic depth ≥ 2. If W is a (good) homological shell of V, then the inequality:*

$$\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$$

holds on their Δ-genera.

Now let us review several known or elementary facts as preliminaries.

Lemma 1.2 *An exact commutative diagram of abelian sheaves are given as follows.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \longrightarrow & 0 \\
 & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\
0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \longrightarrow & 0 \\
 & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_3}\downarrow & & \\
0 & \longrightarrow & C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Set $\rho = g_3 \circ \beta_2 = \gamma_2 \circ g_2$, $\sigma = f_2 \circ \alpha_1 = \beta_1 \circ f_1$, $K = Ker(\rho)$. Then we have a short exact sequence of abelian sheaves :

$$0 \longrightarrow A_1 \xrightarrow{\sigma} K \xrightarrow{\tau_1\oplus\tau_2} C_1 \oplus A_3 \longrightarrow 0,$$

where $\tau_1 = \gamma_1^{-1} \circ (g_2|_K) : K \to C_1, \quad \tau_2 = f_3^{-1} \circ (\beta_2|_K) : K \to A_3$.

Proof. Using the diagonal homomorphism Δ, the difference homomorphism ∇, an isomorphism $(f_1, \alpha_1) : A_1 \to B_1 \times_{B_2} A_2$, and the surjectivity of the homomorphism $(g_2, \beta_2) : B_2 \to C_2 \times_{C_3} B_3 = Ker(\gamma_2 - g_3)$, we obtain an exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A_1 & \xrightarrow{\cong} & B_1 \times_{B_2} A_2 & & 0 & & \\
 & & \sigma\downarrow & & \downarrow(\beta_1,f_2) & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{incl.} & B_2 & \xrightarrow{\rho} & C_3 & \longrightarrow & 0 \\
 & & \exists\downarrow\tau_1\oplus\tau_2 & & \downarrow g_2\oplus\beta_2 & & \downarrow\Delta & & \\
0 & \longrightarrow & C_1\oplus A_3 & \xrightarrow{\gamma_1\oplus f_3} & C_2\oplus B_3 & \xrightarrow{\gamma_2\oplus g_3} & C_3\oplus C_3 & \longrightarrow & 0 \\
 & & & & \gamma_2-g_3\downarrow & & \downarrow\nabla & & \\
 & & & & C_3 & = & C_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Then we apply the snake lemma and get the short exact sequence. ∎

Proposition 1.3 (Corollary of Fulton-Hansen, [3]) *Let Y and Z be projective subvarieties of $P = \mathbb{P}^N(\mathbb{C})$ with $a = \dim Y$ and $b = \dim Z$, respectively. If $a + b > N$, then the intersection scheme $Y \cap Z$ is connected and non-empty.*

Theorem 1.4 (Kleiman, [7], or [6] III. Theorem 10.8) *Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic 0. Let $f : Y \to X$ and $g : Z \to X$ be morphisms of nonsingular varieties Y, Z to X. For any $\sigma \in G(k)$, let Y^σ be Y with the morphism $\sigma \circ f$ to X.*

Then, there is a non-empty open subset $U \subseteq G$ such that for every $\sigma \in U(k)$, $Y^\sigma \times_X Z$ is non-singular and either empty or of dimension exactly,

$$\dim Y + \dim Z - \dim X.$$

Theorem 1.5 (Künneth formula cf. [5] III Théorème (6.7.3)) *Let k be a field, X and Y algebraic schemes over k, E a quasi-coherent O_X-module and F a quasi-coherent O_Y-module. Then*

$$H^p(X \times_k Y, \pi_X^* E \otimes \pi_Y^* F) \cong \bigoplus_{s+t=p} H^s(X, E) \otimes_k H^t(Y, F),$$

where π_X and π_Y denote the projections from $X \times_k Y$ to X and to Y, respectively.

Theorem 1.6 (Theorem 1.7 of [17]) *Let V, $W \subseteq \mathbb{P}^N(\mathbb{C}) = P = Proj(S)$ be closed subschemes, $S = \mathbb{C}[Z_0, \dots, Z_N]$ a polynomial ring over the complex number field $\mathbb{C}$, and $S_+ = (Z_0, \dots, Z_N)S$ the unique homogeneous maximal ideal. Take the homogeneous coordinate rings $R_V = S/\mathbb{I}_V$ and $R_W = S/\mathbb{I}_W$ of V and of W, respectively. The homogeneous ideals $\mathbb{I}_V$ and $\mathbb{I}_W$ are taken as the ones removed S_+-primary components. Assume that:*

(1.6.1) *V and W are locally Cohen-Macaulay and (topologically) equidimensional with $r(V) = codim(V, P) \geq 1$ and $r(W) = codim(W, P) \geq 1$ which satisfy $r(V) + r(W) \leq N - 1$.*

(1.6.2) *V and W meet properly, namely $X = V \cap W$ satisfies $\dim X = N - (r(V) + r(W))$.*

(1.6.3) *There exist non-negative integers k_1 and k_2 with $k_1 + k_2 \geq \dim X - 1$ and*

$$arith.depth(V) \geq r(W) + 2 + k_1$$
$$arith.depth(W) \geq r(V) + 2 + k_2.$$

Then, the subscheme X is of equidimensional and of locally Cohen-Macaulay, which satisfies

$$arith.depth(X) = arith.depth(V) + arith.depth(W) - (N + 1) \geq 2,$$

namely the arithmetic D_2-condition, $Tor_i^S(R_V, R_W) = 0$ $(i \geq 1)$, and both V and W are good homological shells of X.

Notation 1.7 *Let us take integers $e_1 \geq e_2 \geq \ldots \geq e_n \geq 0$, $d = \sum_{i=1}^n e_i \geq 2$, $N = d + n - 1$, and put $B = \mathbb{P}^1(\mathbb{C})$. Then we set a "rational scroll" $S(e_1, e_2, \dots, e_n)$ to be the image $W = Im(f)$ of the birational morphism $f : U = Proj(Sym(\oplus_{i=1}^n O_B(e_i))) \to \mathbb{P}^N(\mathbb{C}) = P$ determined by the tautological line bundle $O_{U/B}(1)$. The variety W is of $\dim(W) = n$ and has $\deg(W) = d$, which is singular if and only if $e_n = 0$. In our case, instead of the variety U, we call the variety W as a rational scroll for simplicity.*

Theorem 1.8 (Green's $K_{p,1}$-theorem [4]) *Let $Z \subseteq \mathbb{P}^N(\mathbb{C}) = P$ be a linearly non-degenerate subvariety of dimension m. For $p = N - m - 1$, if $Tor_p^S(R_Z, S/S_+)_{(p+1)} \neq 0$ and $\deg(Z) \geq p + 4$, then there exists a subvariety $Y \subseteq P$ of minimal degree such that $Z \subseteq Y$ and $\dim(Y) = \dim(Z) + 1$.*

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Let us recall and summarize our strategy for classifying the homological shells of a given projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{C}) = \mathrm{P}roj(S) = P$ which satisfies the arithmetic D_2-condition. By the influence of our results [16] and [18], our knowledge on the movement of homological shells increase massively. Thus some part of our procedure is newly improved here from the one in [15]. We first determine the graded Betti numbers $\{\beta_{q,m}(X)\}$ of X, namely $\beta_{q,m}(X) := \dim_{\mathbb{C}} Tor_q^S(R_X, S/S_+)_{(m)}$. Then a minimal graded S-free resolution $\mathbb{F}_{X,\bullet}$ of the homogeneous coordinate ring $R_X = S/\mathbb{I}_X$ has a form : $F_{X,q} = \oplus_m S(-m)^{\beta_{q,m}(X)} \quad (\forall q \geq 0)$.

Now suppose that a closed subscheme $W \subseteq P$ is a homological shell of X. Then we have inequalities $0 \leq \beta_{q,m}(W) \leq \beta_{q,m}(X)$ $(0 \leq \forall q \leq h(X) = hd_S(R_X) = \max\{q \in \mathbb{Z} | \beta_{q,m}(X) \neq 0\})$, $0 \leq \forall m \leq r(X) := \max\{m \in \mathbb{Z} | \beta_{q,m}(X) \neq 0\})$, otherwise $\beta_{q,m}(X) = \beta_{q,m}(W) = 0$. Hence the number of possible cases of

$\{\beta_{q,m}(W)\}_{q,m}$ is finite. The Hilbert polynomial $A_W(m)$ of the scheme W is calculated from its graded Betti numbers,

$$A_W(m) = \sum_{q,m_0 \geq 0} (-1)^q \beta_{q,m_0}(W) \cdot \dim_{\mathbb{C}} H^0(P, O_P(m - m_0)),$$

which brings us the geometric information on W which includes $\dim(W)$ and $\deg(W)$ as is well-known. The table of all the possible graded Betti numbers of Homological Shells of X is called as "**HS-Betti table**" of X. As a remark, this process can be carried out not only for a closed subscheme which really exists but also for a series of graded Betti numbers $\beta = \{\beta_{q,m}\}_{q,m}$ which might not be realized by a closed subscheme, and bring us a polynomial $A_\beta(m)$, invariants $\dim(\beta)$, $\deg(\beta)$, and $\mathrm{codim}(\beta) = N - \dim(\beta)$.

Moreover, if we have an interest only in good homological shells of X, by choosing $\{\beta_{q,m}(W)\}_{q,m}$ with the condition : $\beta_{q,m}(W) = 0$ for $\forall q > h(X) - (\dim W - \dim X)$, we can easily construct also a table of all the possible graded Betti numbers of Good Homological Shells of X, which will be simply called as "**GHS-Betti table**" of X. Since the goodness condition on the homological shells are distinguished by using the graded Betti numbers, the goodness is kept up to weak shell equivalences.

Remark 1.9 *At least in the case that X is not a variety but a scheme, there is an example of a homological shell which is not a good homological shell (cf. Remark 1.4 of [13]).*

Suppose that we have two homological shells W_1 and W_2 of X. If we have a weak shell equivalence between W_1 and W_2, then we see $\beta_{q,m}(W_1) = \beta_{q,m}(W_2)$ $(\forall q, m)$ by the definition of the weak shell equivalence. The converse is not true in general. In this article, we will give a counter-example that the equalities of the graded Betti numbers : $\beta_{q,m}(W_1) = \beta_{q,m}(W_2)$ $(\forall q, m)$ does not imply the weak equivalence between W_1 and W_2 in general (cf. Theorem 2.1). However, once we fix the graded Betti numbers $\{\beta_{q,m}\}_{q,m}$, the number of weak shell equivalence classes of homological shells X is always finite (cf. [18]).

On the other hand, if there exists an inclusion $W_1 \subseteq W_2$ of the schemes, or equivalently an inclusion of the ideal sheaves $I_{W_1} \supseteq I_{W_2}$ in the structure sheaf O_P of P, then the scheme W_2 is automatically a homological shell of W_1 and $\beta_{q,m}(W_1) \geq \beta_{q,m}(W_2)$ $(\forall q, m)$. However, the converse is not true in general. Namely the inequalities $\beta_{q,m}(W_1) \geq \beta_{q,m}(W_2)$ $(\forall q, m)$ does not always imply the existence of the inclusion $W_1 \subseteq W_2$. For example, in [16], we study the homological shells of a trigonal canonical curve X of $g = 5$ up to weak shell equivalence. Then a quadric hypersurface including X is automatically a homological shell of X. The dimension of the family of quadric hypersurfaces including X is 3. However a quadric hypersurface which includes a given homological shell surfaces W with degree 5 is unique. Here we should make a remark that we do not know the answer to the following problem yet.

Problem 1.10 (Strictness of inclusions) *Suppose that there exist an inclusion $W_1 \subseteq W_2$ of homological shells of X and a weak shell equivalence : $W_1 \sim_w W_1'$. Then does there exist always a homological shell W_2' of X with $W_1' \subseteq W_2'$ and $W_2 \sim_w W_2'$?*

Remark 1.11 *Similar to the problem above, we can ask : if there exist an inclusion $W_1 \subseteq W_2$ of homological shells of X and a weak shell equivalence : $W_2 \sim_w W_2'$, then does there exist always a homological shell W_1' of X with $W_1' \subseteq W_2'$ and $W_1 \sim_w W_1'$? But this is not true in general. A counter-example is given by the same example described above. Namely, let X be a trigonal canonical curve of $g = 5$, W_1 a homological shell surface of $\deg W_1 = 5$, $W_2 = Q$ a unique quadric hypersuface including W_1. Then $rank(Q) = 4$. Take $W_2' = Q'$ to b a quadric hypersurface including X of $rank(Q') \geq 5$* (cf. outside of the closure of the reducible curve D_0 in [16] §3)*, then there is no W_1' what we want.*

Once we have a HS-Betti table or a GHS-Betti table, to show the possibilities of the existence of inclusions, we newly introduce a "Betti diagram" which consists of vertices and of arrows as in the following definition. Each vertex and arrow represent a series of graded Betti numbers and a possibility of the existence of an inclusion, respectively. This Betti diagram is a replacement of our previous concept "maximal inclusion diagram", which might often lead to confusions. This is one of improvements adopted here.

Definition 1.12 (Betti Diagram) *Suppose that a projective subscheme $X \subseteq \mathbb{P}^N(\mathbb{C}) = P$ with codimension $c(X)$ and HS-Betti table or GHS-Betti table of X constructed through the procedure described above are given. For a series of graded Betti numbers $\beta := (\beta_{q,m})_{q,m}$ in the table, we set* $\operatorname{codim}(\beta)$ *to be the codimension of a (virtual) closed subscheme in P with these given graded Betti numbers. Then we construct a diagram by placing these series of graded Betti numbers in the table with a hierarchy depending on their codimensions. The series of graded Betti numbers with the codimension 0, namely $(0, \dots, 0)$ is placed at the first floor. The series of them with the codimension one are placed at the second floor. Similarly, the series of them with the codimension k are placed at the $(k+1)$-th floor. Finally the series of them with the codimension $c(X)$ are placed at the top floor. Then for any two distinct series of graded Betti numbers $\beta := (\beta_{q,m})_{q,m}$ and $\beta' = (\beta'_{q,m})_{q,m}$ in the table with* $\operatorname{codim}(\beta) \geq \operatorname{codim}(\beta')$ *and $\beta_{q,m} \geq \beta'_{q,m}$ $(\forall q, m)$, we draw an arrow $\beta \to \beta'$. Finally we remove the arrow $\beta \to \beta'$ if there exists a series β'' in the table with the arrows $\beta \to \beta''$ and $\beta'' \to \beta'$.*

The diagram constructed by this procedure is called as a Betti diagram*. The Betti diagrams constructed from GHS-Betti tables (resp. from HS-Betti tables) are called also as* GHS-Betti digram (*resp.* HS-Betti digram).

Once we finish constructing a Betti diagram, for each vertex, namely a series of graded Betti numbers, we obtain many geometric information on the (good) homological shell W with these data if it exists. The Δ-genera of the main components with the reduced structures of the (good) homological shell W are also bounded. Applying primary decomposition, the Mayer-Vietoris sequence, and the classification theory by Δ-genus (cf. [2]) to the main components of W with reduced structures, we can carry out the classification of homological shells of X as polarized schemes or projective subscheme if $\deg(W)$ is relatively small.

After the classification of homological shells for each series of graded Betti numbers is done, we can replace each vertex (namely a series of graded Betti numbers) in the Betti diagram, by pairs of the isomorphism classes (cf. which are not weak shell equivalence classes in general,) of explicit projective subschemes which possess the series of graded Betti numbers, and their Δ-genera. The reason why we attach the Δ-genera to the subschemes as the pairs at the vertices in the diagram is coming from Conjecture 1.1. In other words, by drawing this diagram, we can test these conjectures (cf. Remark 1.8 of [15]).

Thus we obtain an "**inclusion diagram**" which is the same concept as the previous one with the name "maximal inclusion diagram"(cf. [15]). Here we should make a notice that one vertex in the Betti diagram, namely one series of graded Betti numbers may be replaced by two or more isomorphism classes of projective subschemes with dividing suitably the vertex and arrows connecting to the vertex into several vertices and arrows.

At the stage of reforming a Betti diagram into an inclusion diagram, for a vertex in the Betti diagram, if we can prove the non-existence of projective scheme with the series of graded Betti numbers corresponding to the vertex, we have to remove the vertex and the arrows connecting to (or from) the vertex, and redirect the arrows around the vertex. To explain redirection of arrows, for example, if we have a path of arrows $\beta \to \beta' \to \beta''$ in the Betti diagram, and have to remove the relay vertex β', we draw an arrow directly $\beta \to \beta''$ if there is no other path of arrows connecting from the vertex β to the vertex β''. In case of checking the existence of other path of arrows connecting from the vertex β to the vertex β'', it is prohibited to follow the arrows in reverse directions. Hence, at the stage of constructing an inclusion diagram, we do not have precise knowledge on the existence of inclusions, weak shell equivalence classes or strict shell equivalence classes of the homological shells which have the same series of graded Betti numbers.

At the final stage, we study the parameter spaces of universal families of homological shells with a given series of graded Betti numbers. By using also Koszul graph maps (cf. [18]), we can determine the weak shell equivalence classes and the strict shell equivalence classes. Depending on these results, we divide or fuse the vertices and arrows in the inclusion diagram. Here we have to make a notice that the Δ-genera are invariants up to weak shell equivalences with assuming the arthmetic D_2-condition on X. For two weak shell equivalence classes $[W_1]$ and $[W_2]$, they are connected by an arrow $[W_1] \to [W_2]$ if there exist two representatives $V_1 \in [W_1]$ and $V_2 \in [W_2]$, and an inclusion $V_1 \subset V_2$. Otherwise, we remove the arrow and redirect it. If for any representative $V_1 \in [W_1]$, there exists a representative $V_2 \in [W_2]$ and an inclusion $V_1 \subset V_2$, then we denote it by $[W_1] \overset{strict}{\to} [W_2]$ and say that the inclusion is **strict**. The diagram constructed by these procedure is called as a "complete inclusion diagram". The concept of complete inclusion diagram is rather similar to our previous concept "strict inclusion diagram", but is slightly different from that a priori. If Problem 1.10 is solved affirmatively, then all the inclusions are automatically strict and the both concepts coincides with each other.

Definition 1.13 (Complete Inclusion Diagram) *Similar to Betti diagrams, a* complete inclusion diagram *for the (good) homological shells of X is a diagram which consists of vertices and arrows. Each vertex represents a weak shell equivalence class of the homological shells of X and its Δ-genus, and an arrow $[W_1] \to [W_2]$ does the situation that there exist two representatives $V_1 \in [W_1]$ and $V_2 \in [W_2]$, and an inclusion $V_1 \subset V_2$. A complete inclusion diagram for the (good) homological shells of X is denoted by $CID(X)$.*

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Now we apply the strategy described above to a specific example handled in this article.

Let V be the image of a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5(\mathbb{C}) = P$. It is well-known that the homogeneous coordinate ring R_V of V is Cohen-Macaulay and has a graded minimal S-free resolution $0 \leftarrow R_V \leftarrow \mathbb{F}_{V,\bullet}$ of the form:

$$\mathbb{F}_{V,\bullet} : 0 \leftarrow F_0 = S \leftarrow F_1 = S(-2)^{\oplus 3} \leftarrow F_2 = S(-3)^{\oplus 2} \leftarrow 0.$$

By taking a sufficiently general projective transformation $\sigma \in \mathrm{PGL}(6, \mathbb{C})$, Proposition 1.3 and Theorem 1.4 show that V and V^σ meet transversely and that the intersection $X = V \cap V^\sigma$ is a connected smooth projective curve. Then we can apply Prof.M.Hashimoto's Remark (cf. [8] Proposition 1.4)), or more generally Theorem 1.6 and see that the graded minimal S-free resolution $\mathbb{F}_{X,\bullet}$ is isomorphic to the complex $\mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{V^\sigma,\bullet}$ and $Tor_q^S(R_V, R_{V^\sigma}) = 0$ for $q \geq 1$, $R_X \cong R_V \otimes R_{V^\sigma}$. In particular, by $\mathbb{F}_{V^\sigma,\bullet} \otimes R_V$, we have an exact sequence:

$$0 \leftarrow R_X \leftarrow R_V \leftarrow R_V(-2)^{\oplus 3} \leftarrow R_V(-3)^{\oplus 2} \leftarrow 0,$$

whose sheafication brings a short exact sequence:

$$0 \leftarrow I_{X/V} \leftarrow \oplus^3 O_V(-2H) \leftarrow \oplus^2 O_V(-3H) \leftarrow 0, \qquad (\#\text{-}1)$$

where $O_V(H) = O_P(1) \otimes O_V$. Since $\mathbb{F}_{X,\bullet} \cong \mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{V^\sigma,\bullet}$, we obtain that $\beta_{0,0}(X) = 1$, $\beta_{1,2}(X) = 6$, $\beta_{2,3}(X) = 4$, $\beta_{2,4}(X) = 9$, $\beta_{3,5}(X) = 12$, $\beta_{4,6}(X) = 4$ otherwise $\beta_{q,m}(X) = 0$. We obtain the Hilbert polynomial $A_X(m) = 9m - 3$ from these graded Betti numbers. Thus $\deg(X) = 9$, $\Delta(X) = 4$, the geometric genus $g(X) = 4$. Now we assume that a closed subscheme $W \subseteq P$ is a good homological shell of X, by using the process of constructing GHS-Betti table described above, we obtain Table 1 except the description on the irreducibility and on the reducedness.

Case No.	codim	$(\beta_{1,2}, \beta_{2,3}, \beta_{2,4}, \beta_{3,5}, \beta_{4,6})$ of W	deg	Δ	irred. / red.
(1)	4	$(6, 4, 9, 12, 4)$	9	4	Y/Y
(2)	3	$(4, 2, 3, 2, 0)$	6	2	?/Y
(3)	2	$(2, 0, 1, 0, 0)$	4	1	Y/Y
(4)	2	$(3, 2, 0, 0, 0)$	3	0	Y/Y
(5)	1	$(1, 0, 0, 0, 0)$	2	0	Y/Y
(6)	0	$(0, 0, 0, 0, 0)$	1	0	Y/Y

Table 1: GHS-Betti table, +the degree and Δ

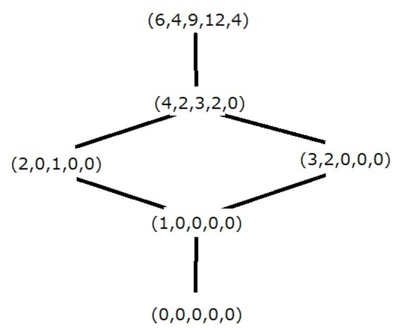

Figure 1: GHS-Betti diagram of $X = V \cap V^\sigma$

Let us study a good homological shell W of X which satisfies one of the cases (1)-(6). The scheme W is automatically arithmetically Cohen-Macaulay by the assumption of goodness, and therefore of equi-dimensional. Obviously, by taking notice on $\beta_{1,2}$, Case(6) $W = P$; Case(5) $W = Q$ an irreducible quadric hypersurface ; Case(3) W is a $(2,2)$-complete intersection ; Case(1) $W = X$.

For Case(3), to see the irreducibility and the reducedness of the scheme W, we take an irreducible component W_0 of W which includes X and give it a reduced structure. Since the curve X is linearly non-degenerate, we see that $4 = \deg(W) \geq \deg((W_0)_{red}) \geq 3$. If $\deg((W_0)_{red}) = 3$, the variety $Y = (W_0)_{red}$ is a variety of minimal degree and $Tor_2^S(R_Y, S/S_+)_{(4)} = 0$. From the inclusions $X \subseteq Y \subseteq W$, the induced map $\mathbb{C} \cong Tor_2^S(R_W, S/S_+)_{(4)} \to Tor_2^S(R_X, S/S_+)_{(4)}$ is a zero map and not an injection, which is a contradiction. Thus $\deg((W_0)_{red}) = 4$ and $W = (W_0)_{red}$. This means that the scheme W is irreducible and reduced. If we take a quadric hypersurface Q of V and a quadric hypersurface Q' of V^σ, then the subscheme $W = Q \cap Q'$ realize Case (3).

For Case (4), by using the same argument for Case (3) and that W is of equi-dimension, it is easy to show that W is a variety and of minimal degree. Applying the well-known classification on the varieties of minimal degree (cf. e.g. [2]), we see that the variety W is a rational scroll of the form $S(e_1, e_2, e_3)$ with $e_1 \geq e_2 \geq e_3 \geq 0$ and $e_1 + e_2 + e_3 = 3$, namely $(e_1, e_2, e_3) = (3, 0, 0)$, or $(2, 1, 0)$, or $(1, 1, 1)$. The Segre 3-folds V and V^σ are the scrolls of the same type $S(1, 1, 1)$ and good homological shells of X. As we show in the next section, V and V^σ are isolate in the universal family of homological shells of X. Namely, e.g. if $W \sim_w V$, then $W = V$. We guess that for Case(4), $W = V$ or $W = V^\sigma$ are the only cases, and have two questions, which are still open.

Question (i) Can other types of scroll, namely $S(3,0,0)$ and $S(2,1,0)$ appear as a good homological shell of X ?

Question (ii) If $W \cong S(1,1,1)$, then $W = V$ or $W = V^\sigma$?

Now we consider Case (2). Let us take the Segre 3-fold V, a quadric hypersurface Q' of V^σ, and set $W = V \cap Q'$. Then the subscheme W realize Case (2).

Next we show that a good homological shell W satisfying Case (2) is reduced. Let us take an irreducible component W_0 of W which includes X. Since X is linearly non-degenerate and $\deg(W) = 6$, the inclusions $X \subseteq (W_0)_{red} \subseteq W$ implies $\deg((W_0)_{red}) \geq 4$ and $W_0 = (W_0)_{red}$. Now we assume that $\deg(W_0) = 4$. Then the surface W_0 is a variety of minimal degree, whose homogeneous coordinate ring R_{W_0} has a 2-linear resolution as a graded minimal S-free resolution. In particular, $Tor_2^S(R_{W_0}, S/S_+)_{(4)} = 0$. On the other hand, by the definition of homological shell, the inclusions of closed subschemes $X \subseteq (W_0)_{red} \subseteq W$ must induce the injective map $Tor_2^S(R_W, S/S_+) \to Tor_2^S(R_X, S/S_+)$. However, this map has a natural factorization $Tor_2^S(R_W, S/S_+) \to Tor_2^S(R_{W_0}, S/S_+) \to Tor_2^S(R_X, S/S_+)$. By taking their deg $= 4$ part, we have an injection : $\mathbb{C}^3 \cong Tor_2^S(R_W, S/S_+)_{(4)} \to Tor_2^S(R_{W_0}, S/S_+)_{(4)} = 0 \to Tor_2^S(R_X, S/S_+)_{(4)}$, which is a contradiction. Thus we have $\deg(W_0) \geq 5$, which implies W is a variety or $W = W_0 \cup L$, where L denotes 2-plane. Hence we see that $W_{red} = W$.

Now we assume moreover that the scheme W is a variety. We apply Green's $K_{p,1}$-theorem, namely Theorem 1.8 to $Z = W$ by setting $p = N - m - 1 = 5 - 2 - 1 = 2$. Since $\deg(W) = 6 \geq p + 4$ and $\dim Tor_2^S(R_W, S/S_+)_{(3)} = \beta_{2,3}(W) = 2 \neq 0$, we have a variety Y of minimal degree such that $W \subseteq Y$ and $\dim(Y) = 3$. Then the variety Y is a good homological shell of X, which fits into Case (4), and $\beta_{1,2}(Y) = 3$. By using $\beta_{1,2}(W) = 4$, we can find a quadric hypersurface Q' satisfying $W \subseteq Q'$ and $Y \not\subseteq Q'$. Then both schemes $W \subseteq Y \cap Q'$ are of pure dimension 2 and of deg $= 6$. Thus we have $W = Y \cap Q'$.

$- * -$

In our previous article [15], we raised the following problem.

Problem 1.14 (Problem 1.12 in [15]) *For a projective subvariety $Y \subseteq \mathbb{P}^N = P$, we set the family of (good) homological shells of Y: $HSF(Y) := \{$(good) homological shell of $Y\} \subseteq \mathfrak{P}(P)$, where $\mathfrak{P}(P)$ denotes the power set of P. Let us assume that a projective subvariety $X \subseteq \mathbb{P}^N = P$ satisfies arithmetic D_2-condition and is represented as the transverse intersection of a projective subvariety $V \subseteq P$ with a hypersurface $D \subseteq P$, then does the following equality hold as the subsets of $\mathfrak{P}(P)$?*

$$HSF(X) \stackrel{?}{=} HSF(V) \cup \{W \cap D \mid W \in HSF(V)\}$$

Inspired by our recent results [16] and [17], we generalize this problem to the following conjecture as one of our working hypotheses.

Working Hypothesis 1.15 (Parallelogram Conjecture) *Let V, $W \subseteq \mathbb{P}^N(\mathbb{C}) = P$ be closed subvarieties of linearly non-degenerate. Set $X = V \cap W$ and assume the same conditions as in Theorem 1.6. Then, the main result of [17] shows that both of V and W are good homological shells of X. Does it always hold the following congruency of the complete inclusion diagrams for good homological shells as directed graphs ?*

$$\Phi : \{CID(V) \times CID(W)\}/\sim_w \xrightarrow{\cong} CID(X), \tag{\#-2}$$

where we consider that the vertices, namely any weak shell equivalence classes $[A] \in CID(V)$ and $[B] \in CID(W)$ are elements of these complete inclusion diagrams, by choosing representatives $A' \in [A]$ and $B' \in [B]$ suitably, which are locally Cohen-Macaulay and of equi-dimensional, and meet properly, the map Φ sends a pair $([A],[B]) \in \{CID(V) \times CID(W)\}$ to $[A' \cap B'] \in CID(X)$. Here we have to prove that we can take representatives $A' \in [A]$ and $B' \in [B]$ which satisfy the conditions above. We have also to show that for any other representatives $A'' \in [A]$ and $B'' \in [B]$, we have a weak shell equivalence $A' \cap B' \sim_w A'' \cap B''$.

Remark 1.16 *Under the circumstances of Working Hypothesis 1.15, we will make several remarks.* (i) *Suppose that the representatives $A' \in [A]$ and $B' \in [B]$ are locally Cohen-Macaulay and of equi-dimensional, and meet properly. Set $a = codim(A', P)$, $b = codim(B', P)$, $k'_1 = arith.depth(A') - b - 2$, $k'_2 = arith.depth(B') - a - 2$. Then, by using the assumption of goodness of homological shells A' and B', we can show $k'_1 \geq 0$ and $k'_2 \geq 0$, $k'_1 + k'_2 \geq \dim(A' \cap B') - 1$. Thus we can apply Theorem 1.6 to A', B' and $Y = A' \cap B'$. The proof of Theorem 1.6 implies the isomorphisms of graded minimal S-free resolutions : $\mathbb{F}_{Y,\bullet} \cong \mathbb{F}_{A',\bullet} \otimes \mathbb{F}_{B',\bullet}$ and $\mathbb{F}_{X,\bullet} \cong \mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}$. Since A' and B' are (good) homological shells of V and of W, respectively, each term of the complex $\mathbb{F}_{A',\bullet}$ and of the complex $\mathbb{F}_{B',\bullet}$ is a direct summand of the corresponding term of the complex $\mathbb{F}_{V,\bullet}$ and of the complex $\mathbb{F}_{W,\bullet}$, respectively. Hence we see that each term of the complex $\mathbb{F}_{A',\bullet} \otimes \mathbb{F}_{B',\bullet}$ is a direct summand of the the corresponding term of the complex $\mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}$. Thus we show that $Y = A' \cap B'$ is a good homological shell of X.* (ii) *It is obvious that $A' \cap B'$ and $A'' \cap B''$ have the same series of graded Betti numbers.*

To see an example of Parallelogram Conjecture holding, we consider (good) homological shells of a non-trigonal canonical curve X of $g = 5$. Since this curve $X \subseteq \mathbb{P}^4(\mathbb{C}) = P$ is a $(2,2,2)$-complete intersection, it is easy to get its complete inclusion diagram C$ID(X)$, which is a diagram in the right hand side of Figure 2. Since $CID(Q_i) = \{[Q_i] \to [P]\}$, Parallelogram Conjecture suggests that for $S_1 = Q_1 \cap Q_2$, $CID(S_1) = \{[S_1] \to [Q_1] = [Q_2] \to [P]\}$, which comes from the parallelogram with 4 vertices S_1, Q_1, P, Q_2 as in Figure 2. Applying a similar process to $X = S_1 \cap Q_3$, we can obtain $CID(X)$ in the right hand side of Figure 2.

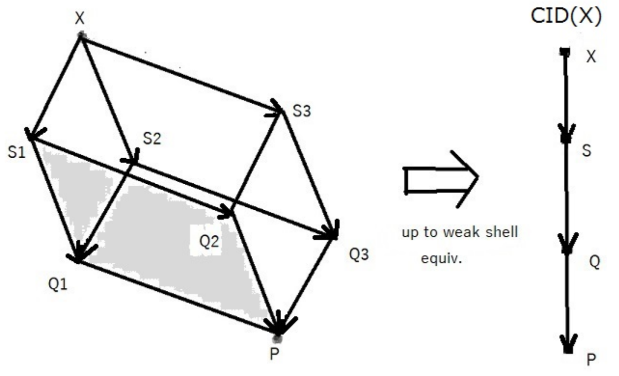

Figure 2: CID of a non-trigonal canonical curve X of $g = 5$

Thus, we have an example where the claim of the congruency (#-2) holds. Hence, even if Parallelogram Conjecture does not hold in general, we still have an interesting problem : Find the criterion for the congruency (#-2) holding through the map Φ.

$- * -$

Now return to our case $X = V \cap V^\sigma$. Let us believe Parallelogram Conjecture temporarily and try to apply this working hypothesis to our case. Since $codim(V, P) = 2$, it is easy to obtain the complete inclusion diagram $\mathrm{C}ID(V) = \{[V] \to [Q] \to [P]\}$, where $[Q]$ denotes the weak shell equivalence class of the quadric hypersurfaces including V, which form also a 2-dimensional linear system of P and has V as the base points. Obviously the complete inclusion diagram of $V' = V^\sigma$ is the similar one $\mathrm{C}ID(V') = \{[V'] \to [Q'] \to [P]\}$. Now if we forget to take the quotient by the weak shell equivalence, then the diagram $\mathrm{C}ID(V) \times \mathrm{C}ID(V')$ have the form as in Figure 3. This figure nearly coincides with our rough classification given above. In other words, Parallelogram Conjecture helps us imagine the result of classification on homological shells.

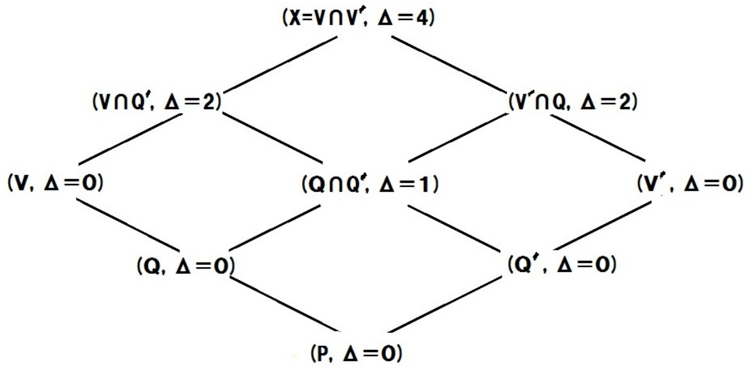

Figure 3: $CID(V) \times CID(V')$ without quotient by $\sim_w$

§2 Infinitesimal deformations of V as a homological shell.

Let us study the tangent space T at V (or at V^σ) of the parameter space of the universal family of homological shells of $X = V \cap V^\sigma \subseteq \mathbb{P}^5(\mathbb{C}) = P$. Take a (good) homological shell W of the curve X with $W \sim_w V$. Then the scheme W is irreducible and reduced, and has $\deg(W) = 3$, $codim(W, P) = 2$, and the Hilbert polynomial : $A_W(m) = A_V(m)$. The scheme W satisfies Case (4) of Table 1.

Next we take a closed subscheme $W' \subseteq P$ which is an embedded deformation of V with including X, and is sufficiently near to V. Then the closed subscheme W' is also a variety, $\deg(W') = 3$, $codim(W', P) = 2$ and $A_{W'}(m) = A_V(m)$. Then, W' is a variety of minimal degree, or more precisely a rational scroll of the form $S(e_1, e_2, e_3)$, where $(e_1, e_2, e_3) = (3, 0, 0)$, or $(2, 1, 0)$, or $(1, 1, 1)$. Its graded Betti numbers are $\beta_{0,0}(W') = 1$, $\beta_{1,2}(W') = 3$, $\beta_{2,3}(W') = 2$ otherwise $\beta_{q,m}(W') = 0$. Hence W' is also a (good) homological shell of X. Thus, locally around V, the parameter space of the universal family of homological shells of X coincides with the parameter space of the Hilbert scheme which parametrizes a closed subscheme W' which includes X and has the Hilbert polynomial $A_{W'}(m) = A_V(m)$. To get the first infinitesimal deformations of V as a homological shell of X is equivalent to get the first infinitesimal embedded deformations of V with including X, which is calculated by $H^0(I_{X/V} \otimes N_{V/P})$, where $N_{V/P}$ denotes the normal sheaf of V in P. This shows that $T = H^0(I_{X/V} \otimes N_{V/P})$ (cf. also, [18]).

To see the bundle structure of the normal sheaf $N_{V/P}$, we set $B = \mathbb{P}^n$, $O_B(H_B) = O_B(1)$, $D = \mathbb{P}^m$, $O_D(H_D) = O_D(1)$, a Segre embedding $\varphi : U := B \times D = \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N = P$ where $N = nm + n + m$, $\varphi^* O_P(1) = \pi_B^* O_B(H_B) \otimes \pi_D^* O_D(H_D) = O_U(H_U)$, projections $\pi_B : B \times D \to B$, $\pi_D : B \times D \to D$. Taking a tensor product of pull backs of two Euler sequences, we have an exact commutative diagram as follows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1 & \longrightarrow & \bigoplus^{n+1} \pi_B^* O_B(-H_B) \otimes \pi_D^*\Omega_D^1 & \longrightarrow & \pi_D^*\Omega_D^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus^{m+1} \pi_B^*\Omega_B^1 \otimes \pi_D^* O_D(-H_D) & \longrightarrow & \bigoplus^{N+1} O_U(-H_U) & \xrightarrow{\beta_1} & \bigoplus^{m+1} \pi_D^* O_D(-H_D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \beta_2 & & \downarrow \alpha_1 & & \\
0 & \longrightarrow & \pi_B^*\Omega_B^1 & \longrightarrow & \bigoplus^{n+1} \pi_B^* O_B(-H_B) & \xrightarrow[\alpha_2]{} & O_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (\#\text{-}3)$$

Putting $\rho = \alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2$, we see that $Ker(\rho) \cong \varphi^*\Omega_P^1 = \Omega_P^1 \otimes O_U$ and $\pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1 \subseteq Ker(\rho)$. Then we apply Lemma 1.2 to the diagram (#-3) and get an short exact sequence :

$$0 \to \pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1 \to \Omega_P^1 \otimes O_U \to \pi_B^*\Omega_B^1 \oplus \pi_D^*\Omega_D^1 \cong \Omega_U^1 \to 0,$$

which brings an isomorphism $N_{U/P}^\vee \cong \pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1$. Hence, the normal sheaf satisfies $N_{U/P} \cong \pi_B^*\Theta_B \otimes \pi_D^*\Theta_D$. The isomorphism $N_{U/P}^\vee \cong \pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1$ can be described explicitly by using the homogeneous coordinates $[s_0 : s_1 : \ldots : s_n]$ of $B = \mathbb{P}^n$, $[t_0 : t_1 : \ldots : t_m]$ of $D = \mathbb{P}^m$, and $[Z_0 : Z_1 : \ldots : Z_N]$ of $P = \mathbb{P}^N$ as follows. Through the Segre embedding φ, for the given indexes $0 \le \alpha, j \le n$, $0 \le \beta, k \le m$, and $0 \le p, q, r, i \le N$, let assume that $Z_p = s_\alpha \otimes t_\beta$, $Z_q = s_j \otimes t_\beta$, $Z_r = s_\alpha \otimes t_k$, and $Z_i = s_j \otimes t_k$. Then the homogeneous ideal $\mathbb{I}_U$ is generated by the quadric equations of the form $Z_p Z_i - Z_q Z_r$. Hence, on the affine set $\varphi(U) \cap D_+(Z_i)$ of P, the bundle $N_{U/P}^\vee$ has the local frame of the forms :

$$\nu(p; q, r/i) = d\left(\left(\frac{Z_p}{Z_i}\right) - \left(\frac{Z_q}{Z_i}\right)\left(\frac{Z_r}{Z_i}\right)\right).$$

Since $\varphi(U) \cap D_+(Z_i) \cong D_+(s_j) \times D_+(t_k)$, the bundle $\pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1$ is free on the affine open $D_+(s_j) \times D_+(t_k)$ with the frame of the forms :

$$\omega(\alpha/j; \beta/k) = d\left(\frac{s_\alpha}{s_j}\right) \otimes d\left(\frac{t_\beta}{t_k}\right).$$

The isomorphism $N_{U/P}^\vee \cong \pi_B^*\Omega_B^1 \otimes \pi_D^*\Omega_D^1$ gives the correspondence between the local section $\nu(p; q, r/i)$ and the local section $\omega(\alpha/j; \beta/k)$ on the affine open $D_+(s_j) \times D_+(t_k)$.

For later use, let us study incidentally the line bundle $det(N_{U/P})$ by applying the splitting principle. For n-bundle $E_1 = \pi_B^*\Theta_B$, and m-bundle $E_2 = \pi_D^*\Theta_D$, taking Chern roots $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$, we have $\Sigma_i u_i = c_1(E_1) = (n+1)\pi_B^* H_B$ and $\Sigma_j v_j = c_1(E_2) = (m+1)\pi_D^* H_D$. Then $c_1(N_{U/P}) = \Sigma_i \Sigma_j (u_i + v_j) = m \cdot c_1(E_1) + n \cdot c_1(E_2)$, which implies that $det(N_{U/P}) \cong \pi_B^* O_B(m(n+1)H_B) \otimes \pi_D^* O_D(n(m+1)H_D)$.

Now let us go back to our case that $n = 1$ and $m = 2$, namely $B = \mathbb{P}^1$, $D = \mathbb{P}^2$, $U = V = \mathbb{P}^1 \times \mathbb{P}^2$. Since $rank(\Omega_D^1) = 2$ and $\wedge^2\Omega_D^1 \cong O_D(-3H_D)$, we have that $\Theta_D \cong \Omega_D^1(3H_D)$, and the normal sheaf satisfies $N_{V/P} \cong \pi_B^* O_B(2H_B) \otimes \pi_D^*\Omega_D^1(3H_D)$. Now, tensoring $N_{V/P}$ to the short exact sequence (#-1), we obtain a short exact sequence :

$$0 \longrightarrow \oplus^2 N_{V/P}(-3H) \longrightarrow \oplus^3 N_{V/P}(-2H) \longrightarrow I_{X/V} \otimes N_{V/P} \longrightarrow 0, \qquad (\#\text{-}4)$$

which induces an exact sequence :

$$\oplus^3 H^0(N_{V/P}(-2H)) \longrightarrow H^0(I_{X/V} \otimes N_{V/P}) \longrightarrow \oplus^2 H^1(N_{V/P}(-3H)). \qquad (\#\text{-}5)$$

Recalling the fact $O_V(H) \cong \pi_B^* O_B(H_B) \otimes \pi_D^* O_D(H_D)$, let us calculate $H^0(N_{V/P}(-2H))$ and $H^1(N_{V/P}(-3H))$ by using Künneth formula (cf. Theorem1.5) and Bott formula (cf. [1]) on $H^p(\mathbb{P}^k, \Omega_{\mathbb{P}^k}^q(m))$.

$$H^0(N_{V/P}(-2H)) \cong H^0(V, \pi_D^*\Omega_D^1(H_D)) \cong H^0(B, O_B) \otimes_{\mathbb{C}} H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(1)) = 0$$

$$\begin{aligned} H^1(N_{V/P}(-3H)) &\cong H^1(\pi_B^* O_B(-H_B) \otimes \pi_D^*\Omega_D^1) \\ &\cong [H^1(O_B(-H_B)) \otimes H^0(\Omega_D^1)] \oplus [H^0(O_B(-H_B)) \otimes H^1(\Omega_D^1)] = 0 \end{aligned}$$

Apply these results to the exact sequence (#-5), we obtain $H^0(I_{X/V} \otimes N_{V/P}) = 0$. Now we summarize our results into the following theorem.

Theorem 2.1 *Let V be a Segre 3-fold, namely an image of a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5 = P$. Take a sufficiently general projective transformation $\sigma \in PGL(6, \mathbb{C})$ such that the Segre 3-folds V and V^σ meet transversely (cf. Theorem 1.4). Then the intersection $X = V \cap V^\sigma$ is a smooth projective curve, which has the invariants : $\deg(X) = 9$, $g(X) = 4$, $\Delta(X) = 4$, and the Hilber polynomial $A_X(m) = 9m - 3$. The Segre 3-folds V and V^σ are good homological shells of X and have the same graded Betti numbers : $\beta_{0,0} = 1$, $\beta_{1,2} = 3$, $\beta_{2,3} = 2$ otherwise $\beta_{q,m} = 0$.*

Then the tangent space T at V or at V^σ of the parameter space of the universal family of homological shells of X is zero. In particular, the Segre 3-folds V and V^σ correspond to isolated points in the parameter space of the universal family of homological shells of X, and they are not weakly shell equivalent to each other.

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