# A generalization of hypersurface cut for constructing new types of homological shells.

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**Abstract**

We supply a new method for constructing homological shells by generalizing hypersurface cut. This method is also a refinement of Prof.M.Hashimoto's Remark (cf. [7] Proposition 1.4). The point is to make the proper intersection of two locally Cohen-Macaulay schemes with relatively large arithmetic depths which are not arithmetically Cohen-Macaulay and to weaken the arithmetic depth of the scheme obtained by the intersection product with keeping to be a homological core. As a corollary, we obtain also the method of cutting a scheme which is not arithmetically Cohen-Macaulay by an arithmetically Cohen-Macaulay scheme which is not a globally complete intersection.
**Keywords**: cut by a scheme with large arithmetic depth, cut by an arthmetically Cohen-Macaulay scheme (=ACMS-cut), homological shell (=pregeometric shell), good homological shell

## §0 Introduction.

Our main interest is to study the "geometric structure" of a given projective embedding of a non-singular projective variety $X$. In other words, once a non-singular projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{C}) = P$ is given, we want to see what kind of projective subscheme $W \subseteq P$ may appear with $X \subseteq W \subseteq P$ and with the Tor-injectivity condition, or with a strong relation of syzygies between $X$ and $W$. Then, the projective schemes $W$ is called a homological shell (=pregeometric shells) of $X$ and $X$ is called a homological core of $W$, respectively (cf. [7] or [8]).

In spite of the facts that the homological shells have the good properties, it is still hard to handle them in general (cf. [9]$\sim$[13]). One of the reasons is that we have only poor methods to construct examples of homological shells and homological cores.

In general, every non-singular projective variety $X$ has a projectively normal embedding, namely an arithmetic $D_2$-embedding (i.e. the depth at the vertex of the affine cone of the embedded variety is greater than or equal to 2). However, we can not expect that it has an arithmetic $D_3$-embedding or more strongly an arithmetically Cohen-Macaulay embedding in general if the dimension is greater than or equal to 2, since there exists a non-singular projective variety with a non-zero irregularity, namely $h^1(O_X) \neq 0$. This means that when we study homological shells in general, we can assume only the arithmetic $D_2$-condition on the homological core but not arithmetic Cohen-Macaulay condition on that.

Except the method of hypersurface cut, an interesting way to construct homological shells through the intersection method was to make a proper intersection of two arithmetically Cohen-Macaulay schemes(cf. Prof.M.Hashimoto's Remark, [7] Proposition 1.4)). However, the homological cores obtained by this method are always arithmetically Cohen-Macaulay. Hence we need to improve this method to construct

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examples of homological shells and homological cores, where the latter objects are not arithmetically Cohen-Macaulay but satisfying the arithmetic $D_2$ condition.

As an application, we can also handle the cutting of non arithmetically Cohen-Macaulay scheme with large arithmetic depth by an arithmetically Cohen-Macaulay scheme. This cutting is called as "arthmetically Cohen-Macaulay scheme cut" (abbrev. ACMS-cut).

## §1 Cut by a scheme with large arithmetic depth.

First we recall several well-known results from the intersection theory or from Projective Geometry, which are very useful also for constructing homological shells (cf. Example 1.11).

**Theorem 1.1 (Fulton-Hansen, [4])** *Let $V$ be a projective variety, $P = \mathbb{P}^N(\mathbb{C})$ an $N$-th projective space, $Q = \prod^m P$ the m-times product of $P$, $f : V \to Q$ a morphism, and $\Delta : P \to Q$ the diagonal morphism. If $\dim f(V) > (m-1)N$, then $f^{-1}(\Delta(P))$ is connected.*

**Corollary 1.2** *Let $Y$ and $Z$ be projective subvarieties of $P = \mathbb{P}^N(\mathbb{C})$ with $a = \dim Y$ and $b = \dim Z$, respectively. If $a + b > N$, then the intersection scheme $Y \cap Z$ is connected and non-empty.*

**Proof.** Apply Theorem 1.1 to the setting : $V = Y \times Z$, $m = 2$, $f = \iota_Y \times \iota_Z$, where $\iota_Y$ and $\iota_Z$ are the inclusion morphisms of $Y$ and of $Z$, respectively. ∎

**Theorem 1.3 (Kleiman, [6], or [5] III. Theorem 10.8)** *Let $X$ be a homogeneous space with group variety $G$ over an algebraically closed field $k$ of characteristic 0. Let $f : Y \to X$ and $g : Z \to X$ be morphisms of nonsingular varieties $Y$, $Z$ to $X$. For any $\sigma \in G(k)$, let $Y^\sigma$ be $Y$ with the morphism $\sigma \circ f$ to $X$.*

*Then, there is a non-empty open subset $U \subseteq G$ such that for every $\sigma \in U(k)$, $Y^\sigma \times_X Z$ is non-singular and either empty or of dimension exactly,*

$$\dim Y + \dim Z - \dim X.$$

Next we recall some results from the commutative ring theory as a preparation of our theorem.

**Theorem 1.4 ([1], [2] Corollary 1.4.14)** *Let $R$ be a Noetherian ring, and $\mathbb{F}_\bullet$ be a bounded complex of finite free $R$-modules with complex length $s = cpx.length(\mathbb{F}_\bullet)$:*

$$\mathbb{F}_\bullet : \quad F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_s \leftarrow 0.$$

*If $\mathbb{F}_\bullet \otimes R_\mathfrak{p}$ is acyclic for all prime ideals $\mathfrak{p}$ with $depth R_\mathfrak{p} < s$, then $\mathbb{F}_\bullet$ is acyclic.*
*Moreover, if the ring $R$ is regular, the condition $depth R_\mathfrak{p} < s$ can be replaced by the condition $ht(\mathfrak{p}) < s$.*

**Corollary 1.5** *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$, $I$ and $J$ ideals of $R$ which satisfy $R/I$ and $R/J$ are Cohen-Macaulay local ring of dimension $s$ and of dimension $t$, respectively. If the local ring $R/(I+J) \cong (R/I) \otimes_R (R/J)$ is of dimension $s + t - n \geq 0$, then*

$$Tor_q^R(R/I, R/J) = 0 \quad (q \geq 1),$$

*and the local ring $R/(I+J)$ is also Cohen-Macaulay.*

**Proof.** Outline of the proof is just carrying out the proof of [7] Proposition 1.4 in a local setting. Namely, taking a minimal free $R$-resolutions $\mathbb{F}_{I,\bullet}$ and $\mathbb{F}_{J,\bullet}$ of the $R$-modules $R/I$ and $R/J$, respectively. Since the local ring $R$ is a regular of dimension $n$, and the local rings $R/I$ and $R/J$ are Cohen-Macaulay of dimension $s$ and of dimension $t$, respectively, the minimal free $R$-resolutions are of the forms $\mathbb{F}_{I,\bullet} = \{F_{I,i}\}_{i=0}^{n-s}$ and $\mathbb{F}_{J,\bullet} = \{F_{J,i}\}_{i=0}^{n-t}$, respectively. Then, the complex $\mathbb{F}_{I,\bullet} \otimes \mathbb{F}_{J,\bullet}$ is also minimal.

Let us show that the complex $\mathbb{F}_{I,\bullet} \otimes \mathbb{F}_{J,\bullet}$ is acyclic by applying Theorem 1.4. Since the length of the complex $\mathbb{F}_{I,\bullet} \otimes \mathbb{F}_{J,\bullet}$ is $2n - s - t$, it is enough to check the exactness of the complex $(\mathbb{F}_{I,\bullet} \otimes \mathbb{F}_{J,\bullet})_{\mathfrak{p}}$ or equivalently $Tor_q^R(R/I, R/J)_{\mathfrak{p}} = 0 \quad (q \geq 1)$ after localizing the prime ideals $\mathfrak{p}$ of $R$ with $ht(\mathfrak{p}) < 2n-s-t$. Moreover, we may assume that $\mathfrak{p} \supseteq I + J$ because the module $Tor_q^R(R/I, R/J)$ has the $R/I$-module structure and the $R/J$-module structure, which implies that the module $Tor_q^R(R/I, R/J)$ is annihilated by the ideal (I+J). By the assumption that the local ring $R/(I+J)$ is of dimension $s+t-n$, the condition $\mathfrak{p} \supseteq I + J$ imply $ht(\mathfrak{p}) \geq n - (s + t - n) = 2n - s - t$. Thus we have nothing to be checked. ∎

**Remark 1.6** *From the view point of Serre's classical intersection theory, the fact of Corollary 1.5 can be considered as a background of the claim [3] Proposition 8.2 (b) when the ambient space is smooth.*

Now let us present a theorem which is a generalization of a (usual) hypersurface cut and also a refinement of Prof.M.Hashimoto's Remark (cf. [7] Proposition 1.4) for handling non arithmetically Cohen-Macaulay schemes.

**Theorem 1.7 (cut by a scheme with large arithmetic depth)** *Let $V$, $W \subseteq \mathbb{P}^N(\mathbb{C}) = P = Proj(S)$ be closed subschemes, $S = \mathbb{C}[Z_0, \ldots, Z_N]$ a polynomial ring over the complex number field $\mathbb{C}$, and $S_+ = (Z_0, \ldots, Z_N)S$ the unique homogeneous maximal ideal. Take the homogeneous coordinate rings $R_V = S/\mathbb{I}_V$ and $R_W = S/\mathbb{I}_W$ of $V$ and of $W$, respectively. The homogeneous ideals $\mathbb{I}_V$ and $\mathbb{I}_W$ are taken as the ones removed $S_+$-primary components. Assume that:*

(1.7.1) *$V$ and $W$ are locally Cohen-Macaulay and (topologically) equidimensional with $r(V) = codim(V, P) \geq 1$ and $r(W) = codim(W, P) \geq 1$ which satisfy $r(V) + r(W) \leq N - 1$.*

(1.7.2) *$V$ and $W$ meet properly, namely $X = V \cap W$ satisfies $\dim X = N - (r(V) + r(W))$.*

(1.7.3) *There exist non-negative integers $k_1$ and $k_2$ with $k_1 + k_2 \geq \dim X - 1$ and*

$$arith.depth(V) \geq r(W) + 2 + k_1$$
$$arith.depth(W) \geq r(V) + 2 + k_2.$$

*Then, the subscheme $X$ is of equidimensional and of locally Cohen-Macaulay, which satisfies*

$$arith.depth(X) = arith.depth(V) + arith.depth(W) - (N + 1) \geq 2,$$

*namely the arithmetic $D_2$-condition, $Tor_i^S(R_V, R_W) = 0$ $(i \geq 1)$, and both $V$ and $W$ are good homological shells of $X$. Moreover, the subscheme $X$ is linearly non-degenerate if and only if the both subschemes $V$ and $W$ are linearly non-degenerate. If the linear non-degeneracy holds and both $V$ and $W$ are varieties, we have*

$$\Delta(X) = \Delta(V) + \Delta(W) + (\deg(V) - 1)(\deg(W) - 1) > \Delta(V) + \Delta(W) \geq max\{\Delta(V), \Delta(W)\},$$

*where their polarizations are naturally induced by $O_P(1)$.*

**Proof.** The strategy of our proof is essentially the same as the one for Corollary 1.5 except handling the $S_+$-primary component carefully.

Take minimal graded $S$-free resolutions $\mathbb{F}_{V,\bullet} = \{F_{V,i}\}_{i=0}^{\ell(V)}$ and $\mathbb{F}_{W,\bullet} = \{F_{W,i}\}_{i=0}^{\ell(W)}$ of the homogeneous coordinate rings $R_V$ and $R_W$, respectively. Then, the lengths of complexes $\mathbb{F}_{V,\bullet}$ and $\mathbb{F}_{W,\bullet}$ are estimated by

$$\begin{aligned}\ell(V) = N+1-\mathrm{a}rith.depth(V) \le N+1-(r(W)+2)-k_1\\ \ell(W) = N+1-\mathrm{a}rith.depth(W) \le N+1-(r(V)+2)-k_2,\end{aligned}$$

respectively. Obviously the complex $\mathbb{F}_{V,\bullet}\otimes_S\mathbb{F}_{W,\bullet}$ is also minimal. The length of the complex $\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}$ satisfies

$$\begin{aligned} \mathrm{c}px.length(\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}) &= \ell(V)+\ell(W)\\ &= 2N-2-(r(V)+r(W))-(k_1+k_2)\\ &\le 2N-2-(r(V)+r(W))-\dim X+1\\ &= 2N-2-(r(V)+r(W))-N+(r(V)+r(W))+1\\ &= N-1.\end{aligned}$$

Let us show that the complex $\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}$ is acyclic by using Theorem 1.4. Since we have the inequality : $\mathrm{c}px.length(\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}) \le N-1$, we have only to show that the complex $(\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet})_{\mathfrak{p}}$ is acyclic, or equivalently $\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{p}} = 0$ $(i\ge 1)$ for any prime ideal $\mathfrak{p}\in \mathrm{S}pec(S) = \mathbb{A}^{N+1}$ with $\mathrm{h}t(\mathfrak{p}) < N-1$.

Take the affine cones $C(V)$, $C(W)$, $C(X)$ in $\mathrm{S}pec(S) = \mathbb{A}^{N+1}$ of the projective subschemes $V$, $W$, $X$ of $\mathrm{P}roj(S) = \mathbb{P}^N$. Take also the affine coordinate rings $R_{C(V)}$, $R_{C(W)}$, $R_{C(X)}$ and $R_{C(V)\cap C(W)} \cong R_{C(V)}\otimes R_{C(W)}$ of the affine closed subschemes $C(V)$, $C(W)$, $C(X)$ and $C(V)\cap C(W)$, respectively. Obviously we have $R_V = R_{C(V)}$, $R_W = R_{C(W)}$, $R_X = R_{C(X)}$ by ignoring the gradings of the former homogeneous coordinate rings, and the canonical surjective ring homomorphism : $R_{C(V)\cap C(W)} \to R_{C(X)}$ which is isomorphic except an $S_+$ primary component. Since $\dim(X) = N-(r(V)+r(W)) \ge 1$, we have $\dim(C(V)\cap C(W)) = \dim(C(X)) = \dim(X)+1 =\ge 2$. Then

$$\begin{aligned}\dim(C(V)\cap C(W)) &= N-(r(V)+r(W))+1\\ &= (N-r(V)+1)+(N-r(W)+1)-(N+1)\\ &= \dim(C(V))+\dim(C(W))-\dim(\mathbb{A}^{N+1}),\end{aligned}$$

which means that the affine closed subschemes $C(V)$ and $C(W)$ meet properly in the affine space $\mathbb{A}^{N+1}$.

The $S$-modules $\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{p}}$ $(i\ge 1)$ have the $R_V$-module structure and $R_W$-module structure through the $S$-module structure, namely $\mathrm{S}upp(\mathrm{T}or_i^S(R_V,R_W)) \subseteq C(V)\cap C(W)$.

Hence it is enough to show that $\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{p}} = 0$ $(i\ge 1)$ for any prime ideal $\mathfrak{p}\in C(V)\cap C(W)$ with $\mathrm{h}t(\mathfrak{p}) < N-1$. Then $V(\mathfrak{p}) \subseteq C(V)\cap C(W)$ and $\dim V(\mathfrak{p}) \ge 3$. This implies that there is a closed point $\mathfrak{m}\in V(\mathfrak{p})$ which is different from the vertex point $S_+ \in C(V)\cap C(W)$. Since the projective subschemes $V$ and $W$ are locally Cohen-Macaulay, the affine subschemes $C(V)$ and $C(W)$ are also locally Cohen-Macaulay outside the vertex point $S_+$. By applying Corollary 1.5 with a fact that the affine closed subschemes $C(V)$ and $C(W)$ meet properly in the affine space $\mathbb{A}^{N+1}$, we see that $\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{m}} \cong \mathrm{T}or_i^{S_{\mathfrak{m}}}(R_{V,\mathfrak{m}},R_{W,\mathfrak{m}}) = 0$ $(i\ge 1)$. Thus, for the generalization $\mathfrak{p}$ of $\mathfrak{m}$, $\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{p}} \cong (\mathrm{T}or_i^S(R_V,R_W)_{\mathfrak{m}})_{\mathfrak{p}} = 0$ $(i\ge 1)$.

Now we know that the complex $\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}$ is acyclic and $\mathrm{c}px.length(\mathbb{F}_{V,\bullet}\otimes\mathbb{F}_{W,\bullet}) \le N-1$. Hence the graded ring $R_V\otimes R_W \cong S/(\mathbb{I}_V+\mathbb{I}_W)$ has the arithmetic depth $\ge 2$, namely the graded ring $R_V\otimes R_W$ does not have $S_+$ as an associated prime. Thus we see that $R_V\otimes R_W \cong R_X$ and $\mathrm{a}rith.depth(X) \ge 2$.

This show that the homogeneous coordinate ring $R_X$ has the acyclic complex $\mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}$ as its minimal graded $S$-free resolution and its arithmetic depth is exactly computed by

$$\begin{aligned} \mathrm{a}rith.depth(X) &= (N+1) - \mathrm{c}px.length(\mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}) \\ &= (N+1) - (\ell(V) + \ell(W)) \\ &= (N+1) - (N+1-\mathrm{a}rith.depth(V)) - (N+1-\mathrm{a}rith.depth(W)) \\ &= \mathrm{a}rith.depth(V) + \mathrm{a}rith.depth(W) - (N+1). \end{aligned}$$

Let us recall the fact that $F_{V,0} = F_{W,0} = S$, which induces the complex homomorphisms : $1 \otimes F_{W,0} : \mathbb{F}_{V,\bullet} \hookrightarrow \mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}$ and $F_{V,0} \otimes 1 : \mathbb{F}_{W,\bullet} \hookrightarrow \mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}$. It should be noticed that each term of the minimal acyclic complex in the left hand side is a direct summand of the corresponding term of the one in the right hand side through these complex homomorphisms. These facts easily show the T$or$-injectivities, namely that the induced maps $\mathrm{T}or_i^S(R_V, S/S_+) \to \mathrm{T}or_i^S(R_X, S/S_+)$ $(i \geq 1)$ and $\mathrm{T}or_i^S(R_W, S/S_+) \to \mathrm{T}or_i^S(R_X, S/S_+)$ $(i \geq 1)$ are injective. In other words, $V$ and $W$ are homological shells of $X$. Let us show that these homological shell $V$ and $W$ of $X$ are good. $\dim V - \dim X = (N - r(V)) - (N - r(V) - r(W)) = r(W)$. On the other hand, $\mathrm{h}d_S(R_X) - \mathrm{h}d_S(R_V) = \mathrm{c}px.length(\mathbb{F}_{V,\bullet} \otimes \mathbb{F}_{W,\bullet}) - \ell(V) = \ell(W) = (N+1) - \mathrm{a}rith.depth(W) \geq N+1-(\dim W + 1) = r(W)$. Thus we see that $\mathrm{h}d_S(R_X) - \mathrm{h}d_S(R_V) \geq \dim V - \dim X$, which means that the homological shell $V$ of $X$ is good. By the similar calculation, we can see also that the homological shell $W$ of $X$ is good.

On the claim of linear non-degeneracy, we have only to take a notice on the fact that the spaces $\mathrm{T}or_1^S(R_V, S/S_+)$ and $\mathrm{T}or_1^S(R_W, S/S_+)$ can be naturally considered as the subspaces of $\mathrm{T}or_1^S(R_X, S/S_+)$ via T$or$-injectivity condition of homological shells, and $\mathrm{T}or_1^S(R_V, S/S_+)_{(1)} + \mathrm{T}or_1^S(R_W, S/S_+)_{(1)} = \mathrm{T}or_1^S(R_X, S/S_+)_{(1)}$ and the space $\mathrm{T}or_1^S(R_\Box, S/S_+)_{(1)}$ represents the number of linearly independent linear equations including the scheme $\Box = X,\ V,\ W$, where the subscript (1) means the degree 1-part of $\mathrm{T}or_1^S(\ -\ , S/S_+)$.

To see the equality on $\Delta$-genera, we have only to use the definition of $\Delta$-genus and the facts that $\deg X = \deg V \cdot \deg W$ and $h^0(O_X(1)) = h^0(O_V(1)) = h^0(O_W(1)) = N+1$, which is obtained from the linear non-degeneracy, the arithmetic $D_2$-conditions of $V$, $W$, and $X$.

∎

**Remark 1.8** *Under the circumstances of Theorem 1.7, if the equalities* $arith.depth(V) = r(W) + 2 + k_1$, $arith.depth(W) = r(V) + 2 + k_2$ *and* $k_1 + k_2 = \dim X - 1$ *hold, then exactly* $arith.depth(X) = 2$.

**Remark 1.9** *To see the meaning of the inequalities* (1.7.3) *in Theorem 1.7, we take positive integers* $n >> t > 0$. *To avoid the range predicted by Hartshorne C.I. conjecture, we set* $N = 6n$, $r(V) = r(W) = 2n + t$, $k_1 = k_2 = n - t$, *then* $\dim V = \dim W = 4n - t$ *and* $\dim X = 2n - 2t$. *Assume the equalities in* (1.7.3). *Then,* $\dim C(V) - arith.depth(V) = \dim C(W) - arith.depth(W) = n - t - 1 >> 0$, *namely, for* $V$ *or for* $W$, *the gap between the dimension of the affine cone and arithmetic depth is relatively large.*

Let us simplify the numerical conditions of Theorem 1.7 and give a new method of "cut by an arithmetically Cohen-Macaulay scheme" (abbrev. "ACMS-cut") for constructing new types of homological shells and homological cores.

**Corollary 1.10 (cut by an arithmetically Cohen-Macaulay scheme)** *Let* $V,\ W \subseteq \mathbb{P}^N(\mathbb{C}) = P = Proj(S)$ *be closed subschemes,* $S = \mathbb{C}[Z_0, \ldots, Z_N]$ *a polynomial ring over the complex number field* $\mathbb{C}$, $S_+ = (Z_0, \ldots, Z_N)S$ *the unique homogeneous maximal ideal. Assume that:*

(1.10.1) $V$ *is arithmetically Cohen-Macaulay with* $codim(V, P) = r \geq 1$, (1.10.2) *$W$ is locally Cohen-Macaulay and of equidimensional with* $\dim W = m$*, whch satisfies the arithmetic $D_{r+2}$-condition, namely arith.depth$(W) \geq r+2$ (N.B. hence $m \geq r+1$),*

(1.10.3) *$V$ and $W$ meet properly, namely $X = V \cap W$ satisfies* $\dim X = m - r$.

*Then, the subscheme $X$ is of equidimensional and of locally Cohen-Macaulay, which satisfies*

$$arith.depth(X) = arith.depth(W) - r \geq 2,$$

*namely the closed subscheme $X$ satisfies the arithmetic $D_2$-condition, $Tor_i^S(R_V, R_W) = 0$ $(i \geq 1)$, and both $V$ and $W$ are good homological shells of $X$. Moreover, the subscheme $X$ is linearly non-degenerate if and only if the both subschemes $V$ and $W$ are linearly non-degenerate. If the linear non-degeneracy holds and both $V$ and $W$ are varieties, we have*

$$\Delta(X) = \Delta(V) + \Delta(W) + (\deg(V) - 1)(\deg(W) - 1) > \Delta(V) + \Delta(W) \geq max\{\Delta(V), \Delta(W)\},$$

*where their polarizations are naturally induced by $O_P(1)$.*

**Proof.** Under the circumstances of Theorem 1.7, it is enough to set $r(V) = r$, $r(W) = N - m$, $k_1 = m - r - 1$, and $k_2 = 0$. ∎

Let us apply Corollary 1.10 and present an example whose homological core is a non-singular projective variety and is not arithmetically Cohen-Macaulay.

**Example 1.11** *Let us take a sufficiently large integer $n$ and a smooth hypersurface $Y \subseteq \mathbb{P}^n = M$ of degree $d$ with $d \geq n+1$. Then it is easy to see that $h^k(O_Y(mH_1)) = 0$ ( $1 \leq k \leq n-2$, $m \in \mathbb{Z}$ ), where $O_Y(H_1)$ denotes the ample line bundle on $Y$ induced from the ample tautological line bundle $O_M(1) = O_M(H_1)$ of $\mathbb{P}^n = M$. In particular, we have $h^k(O_Y) = 0$ $(1 \leq k \leq n-2)$. By Hodge symmetry, we see that $h^0(\Omega_Y^k) = 0$ ( $1 \leq k \leq n-2$ ). On the other hand $\Omega_Y^{n-1} \cong K_Y \cong O_Y((d-n-1)H_1)$, which implies that $h^0(\Omega_Y^{n-1}) \neq 0$.*

*Let us put $B = \mathbb{P}^1$, $O_B(H_2) = O_{\mathbb{P}^1}(1)$, $W = Y \times B$, projections $\pi_1 : W = Y \times B \to Y$, $\pi_2 : W = Y \times B \to B$ and give an embedding $W = Y \times B \hookrightarrow M \times B \hookrightarrow \mathbb{P}^{2n+1} = P$ through the Segre embedding. Then we put $O_W(H) := O_P(1) \otimes O_W$ and see that $O_W(H) = \pi_1^* O_Y(H_1) \otimes \pi_2^* O_B(H_2)$, or simply $H = H_1 + H_2$ with abbreviations. By using $H^0(W, O_W(mH)) \cong H^0(Y, O_Y(mH_1)) \otimes H^0(O_B(mH_2))$ and the surjectivity $H^0(M, O_M(mH_1)) \to H^0(Y, O_Y(mH_1))$, it is easy to see that the projective submanifold $W$ satisfies the arithmetic $D_2$-condition. Since $\Omega_W^1 \cong \pi_1^*\Omega_Y^1 \oplus \pi_2^* O_B(-2H_2)$, for an integer $k$ with $1 \leq k \leq n-1$, $\Omega_W^k \cong \pi_1^*\Omega_Y^k \oplus \pi_1^*\Omega_Y^{k-1} \otimes \pi_2^* O_B(-2H_2)$, which imply $h^0(W, \Omega_W^k) = h^0(Y, \Omega_Y^k)$. Then we see that $h^0(W, \Omega_W^k) = 0$ for ( $1 \leq k \leq n-2$ ) and $h^0(W, \Omega_W^{n-1}) \neq 0$. Now by applying Hodge symmetry again, we have $h^k(W, O_W) = 0$ for ( $1 \leq k \leq n-2$ ) and $h^{n-1}(W, O_W) \neq 0$.*

*Let us show that $H^k(W, O_W(mH)) = 0$ for $(1 \leq k \leq n-2)$. Since the divisor $H$ is ample, by Kodaira vanishing, $H^k(W, O_W(mH)) = 0$ for $(1 \leq k \leq n-2)$ and $m < 0$. If $m > 0$, then $H^k(W, O_W(mH)) \cong H^k(Y, O_Y(mH_1)) \otimes H^0(B, O_B(mH_2)) \oplus H^{k-1}(Y, O_Y(mH_1)) \otimes H^1(B, O_B(mH_2)) = 0$. Hence arith.depth$(W) = n < \dim(C(W)) = n+1$. Thus the projective submanifold $W \subseteq P$ is not arithmetically Cohen-Macaulay.*

*Next we consider another Segre embedding $Z = \mathbb{P}^2 \times \mathbb{P}^1 \subseteq \mathbb{P}^5$. Take $(2n-4)$-multiple cone $C^{(2n-4)}(Z) \subseteq \mathbb{P}^{2n+1} = P$ which is obviously arithmetically Cohen-Macaulay. Then $\dim C^{(2n-4)}(Z) = 2n-1$ and $\dim Sing(C^{(2n-4)}(Z)) = 2n-5$. Apply Bertini's theorem, we take $(n-5)$-times hypersurface cut $V = C^{(2n-4)}(Z) \cap D_1 \cap \ldots \cap D_{n-5}$ of $C^{(2n-4)}(Z)$ by general hypersurfaces $D_1, \ldots, D_{n-5}$ of degree greater than or equal to 2, and may assume that the scheme $V$ is irreducible and reduced, and moreover*

*$\dim V = n+4$ and $\dim Sing(V) = n$. We set $V^\circ := V - Sing(V)$. Now we consider the projective space $P = \mathbb{P}^{2n+1}$ to be a homogeneous space with an action of $G = PGL(2n+2,\mathbb{C}) = GL(2n+2,\mathbb{C})/\mathbb{C}^\times$ and apply Theorem 1.3. There exists a non-empty Zariski open subset $U_1 \subseteq G$ such that for any $\sigma \in U_1(\mathbb{C})$, we have $Sing(V)^\sigma \cap W = \phi$. There exists also a non-empty Zariski open subset $U_2 \subseteq G$ such that for any $\sigma \in U_2(\mathbb{C})$, the intersection $(V^\circ)^\sigma \cap W$ is a smooth manifold of dimension 3. Since $G = PGL(2n+2,\mathbb{C})$ is an irreducible variety, the intersection $U = U_1 \cap U_2$ is also a non-empty Zariski open subset of $G$. Hence for any $\sigma \in U(\mathbb{C})$, $X := V^\sigma \cap W = (V^\circ)^\sigma \cap W$ is smooth of dimension 3. By Corollary 1.2, $X := V^\sigma \cap W$ is connected with respect to the classical topology. Thus $X$ is a non-singular projective variety of dimension 3. We apply Corollary 1.10 and see that $arith.depth(X) = 3 < \dim C(X) = 4$, which is not arithmetically Cohen-Macaulay.*

*From the construction of $X$ above, we see that*

$$\begin{aligned}
&H^0(P, O_P(mH)) \overset{surj.}{\to} H^0(X, O_X(mH)) \quad (\forall\, m \in \mathbb{Z})\\
&H^1(X, O_X(mH)) = 0 \quad (\forall\, m \in \mathbb{Z})\\
&H^2(X, O_X(m_0H)) \neq 0 \quad (\exists\, m_0 \in \mathbb{Z}).
\end{aligned}$$

*Corollary 1.10 shows that $V^\sigma$, $W$ and therefore $C^{(2n-4)}(Z)^\sigma$ are good homological shells of $X$.*

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