Homological shells of a canonical curve of genus 6 (II) (plane quintic type)

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Abstract

We continue to classify good homological shells of a canonical curve X with genus $g = 6$. Here we consider the case that the canonical curve X is of plane quintic type. In this case, the classification of the good homological shells is more complicated than in the generic case already handled in [23], and is not completed yet. However, also in this case, we can verify the inequality on Δ-genera of the good homological shells, which is predicted by our Δ-genus inequality conjecture in [14]. The summary of this result is given by the maximal inclusion diagram (#-2).

Keywords: (good) homological shell, pregeometric shell, canonical curve, plane quintic, genus 6, Δ-genus inequality conjecture

§0 Introduction.

All the problems in [14] and in [19] arose from our faith that there must exist a geometry of projective embeddings which reflects the internal geometry of projective varieties. As a main tool for exploring into the geometry of projective embeddings, we have a special interest in intermediate ambient schemes which satisfy certain good conditions from the view point of syzygies for the given embedded variety. Those intermediate ambient schemes are called as homological shells (previously called as "pregeometric shells"), whose precise definition is given in Definition 1.1.

Among the problems in [14] and [19], the most important and fundamental one is Conjecture (1.3) including Δ-genus inequality conjecture (1.3.2). In our articles [15] - [17], and [20] - [23], through the classification of homological shells of a given embedded projective variety, we found several examples of homological shells, which also give evidences for this conjectures and bring a new conjecture (cf. Conjecture 1.8 in [23]). Moreover, some of these examples brought us homological shells of a new type except the typical shells, namely original models of homological shells which motivated our research.

On the other hand, we found that the arithmetic Cohen-Macaulay property or equidimensionality is not inherited from the homological cores to their homological shells in general. In the classification of homological shells of a canonical curve with $g = 6$, to avoid technical nuisances caused by this defect of the homological shells, we restrict ourselves to study "good" homological shells (cf. (1.1.2)) only. Obviously, good homological shells always inherit the arithmetic Cohen-Macaulay property from their homological cores. As a remark, an advantage of the concept "good homological shell" comes from the fact that all the typical shells are always good homological shells but not arithmetically Cohen-Macaulay in general when the homological cores are not arithmetically Cohen-Macaulay.

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This article is an additional part of the series of our papers mentioned above. Here, we take a canonical curve of plane quintic type with genus $g = 6$ as the embedded projective variety and study its good homological shells. For these good homological shells, we can confirm monotonously Δ-decreasing, namely Δ-genus inequality conjecture (1.3.2) holding among them. We summarize this result by drawing a maximal inclusion diagram as in (#-2).

It should be noticed that a maximal inclusion diagram guarantees the non-existence of an inclusion by a vacant space without an arrow. In other words, an arrow in a maximal inclusion diagram shows only the possibility of the existence of an inclusion. Since the homological shells move in their families, we often have to put up with drawing a maximal inclusion diagram instead of a diagram which describes inclusions precisely. Moreover, maximal inclusion diagrams are very useful to confirm monotonously Δ-decreasing quickly.

In this article, we use successively the notation and conventions in [5], [20], and [23] without mention.

§1 Preliminaries.

Let us recall our key concept for studying the geometric structures of projective embeddings. The concept "homological shell" was introduced first in [13]. We can find many good actual examples of this concept in a number of classical works in Complex Projective Geometry such as [8], [9], [11], [3] and so on.

Definition 1.1 (shells and cores) *Take a polynomial ring $S := \mathbb{C}[Z_0, \dots Z_N]$ of $(N + 1)$-variables over the complex number field $\mathbb{C}$ with the usual grading, and its maximal homogeneous ideal $S_+ := (Z_0, \dots Z_N)S$. Let V and W be closed sub*<u>*schemes*</u> *of $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ which satisfy $V \subseteq W$ (namely the inclusion of the defining ideal sheaves: $I_V \supseteq I_W$ in the structure sheaf O_P of P; In this case, the subscheme W is called simply an* intermediate ambient scheme *of V).*

(1.1.1) *If the natural map:*

$$\mu_q : Tor^S_q(R_W, S/S_+) \to Tor^S_q(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ (abbr. "global Tor injectivity condition"*), we say that W is a* homological shell *(abbr.* H-shell*) of V and that V is a* homological core *(abbr.* H-core*) of W, where $R_W := S/\mathbb{I}_W$ and $R_V := S/\mathbb{I}_V$ denote the homogeneous coordinate rings of W and of V, respectively, and $\mathbb{I}_W := \oplus_m H^0(P, I_W(m))$, $\mathbb{I}_V := \oplus_m H^0(P, I_V(m))$.*

(1.1.2) *Now we set $r_0 := \dim(W) - \dim(V)$. Assume that the scheme W is a homological shell of V and that for every integer $q \geq 0$, we have $Tor^S_{q-r_0}(R_W, S/S_+) = 0$ if $Tor^S_q(R_V, S/S_+) = 0$. Then we say that the scheme W is a* good *homological shell of V and the scheme V is a* good *homological core of W.*

For the subscheme V, the total space P and V itself are called as trivial *(good) homological shells.*

Remark 1.2 (i) *Homological shell defined in (1.1.1) above was called as* pregeometric shell *or* PG-shell *in our several previous works after we introduced this concept in [13].*

(ii) *The condition "good" in (1.1.2) is the same to say the inequality on the homological dimensions $hd_S(R_W) \leq hd_S(R_V) - r_0$, or equivalently $arith.depth(W) \geq arith.depth(V) + r_0$ by Auslander-Buchsbaum formula. Thus, supposing that the scheme V is arithmetically Cohen-Macaulay and that the scheme W is a homological shell of V, it is obvious that the scheme W is arithmetically Cohen-Macaulay if and only if the homological shell W is* good.

(iii) *Homological shell is not always good. For example, see the example given by Remark (1.4) in [22]. This example also shows that a closed subscheme which has an arithmetically Cohen-Macaulay homological core is not always arithmetically Cohen-Macaulay nor equidimensional.*

Let us recall our two fundamental conjectures on homological shells from [14] and [15].

Conjecture 1.3 *Let $P = \mathbb{P}^N(\mathbb{C})$ be an N-th projective space with the tautological ample line bundle $O_P(1) = O_P(H)$ and $V \subseteq W \subseteq P$ its closed subschemes.*

(1.3.1) *Assume that the scheme V is a variety, namely reduced and irreducible and that the closed subscheme W is a homological shell of V. Then the subscheme W is also a variety.*

(1.3.2) [**Δ-genus inequality conjecture**] *Suppose that the subscheme V is arithmetically D_2, namely its arithmetic depth ≥ 2. If W is a homological shell of V, then the inequality:*

$$\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$$

holds on their Δ-genera.

Remark 1.4 *For a polarized* scheme (V, L), *namely a pair of a projective scheme V and an ample invertible sheaf L on it, its Δ-genus is defined by $\Delta(V, L) := dim(V) + \deg(L) - h^0(V, L)$, where $\deg(L)$ is defined by using its Hilbert polynomial. On the general theory of Δ-genus for a polarized* variety (V, L), *see [3]. In our previous papers, we assumed that both the schemes V and W are varieties in the conjecture (1.3.2) since Δ-genus is usually defined for a pair of a variety and an ample line bundle on it. However, it is convenient to generalize the statement of the conjecture (1.3.2) for closed schemes from the technical view point since the two conjectures can be handled independently. For additional information on these two conjectures, see §1 of [17].*

Remark 1.5 *In the conjecture (1.3.2), the assumption of arithmetic D_2-condition is essential. Without this assumption, we can make a counter-example as follows. First we take an arithmetically non-Cohen-Macaulay smooth rational quartic curve, namely*

$$V = \mathbb{P}^1 \ni [s_0 : s_1] \longmapsto [Z_0 : Z_1 : Z_2 : Z_3] = [s_0^4 : s_0^3 s_1 : s_0 s_1^3 : s_1^4] \in \mathbb{P}^3.$$

Then, the homogeneous ideal $\mathbb{I}_V$ has a system of minimal generators : $\{Z_0Z_3 - Z_1Z_2,\ Z_1^3 - Z_0^2Z_2,\ Z_2^3 - Z_1Z_3^2\ Z_0Z_2^2 - Z_1^2Z_3\}$. Let us set $W := \{Z_1^3 - Z_0^2Z_2 = 0\}$. Obviously the cubic surface W is a homological shell of the curve V. On the other hand, $\Delta(W, O_W(1)) = 1 > 0 = \Delta(V, O_V(1))$ since $(V, O_V(1)) \cong (\mathbb{P}^1, O_{\mathbb{P}^1}(4))$.

Remark 1.6 *For a polarized variety (V, L), then $\Delta(V, L) \geq 0$ by (4.2) Theorem of [3]. However, for a polarized scheme (V, L), it may happen that $\Delta(V, L) < 0$. To make an example, let us take a large enough integer N, 2-plane $H \subseteq \mathbb{P}^N = P$, a conic $Q \subseteq H$, distinct closed n-points $p_i \in \mathbb{P}^N$ $(i = 1, 2, \ldots n)$ in general position with respect to the 2-plane H, and set a closed scheme $V := Q \cup (\bigcup_{i=1}^n \{p_i\})$ and an ample invertible sheaf $L := O_P(1)|_V$. Then $\Delta(V, L) = -n < 0$. Even if we demand the connectedness of the scheme V, taking a 3-plane H, a quadric surface $Q \subseteq H$, a closed point $p_0 \in Q$, and distinct closed n-points $p_i \in \mathbb{P}^N$ $(i = 1, 2, \ldots n)$ in general position with respect to the 3-plane H, it is enough to set $V := Q \cup (\bigcup_{i=1}^n \ell(p_i, p_0))$, where $\ell(p_i, p_0)$ denotes the line joining the points p_i and p_0.*

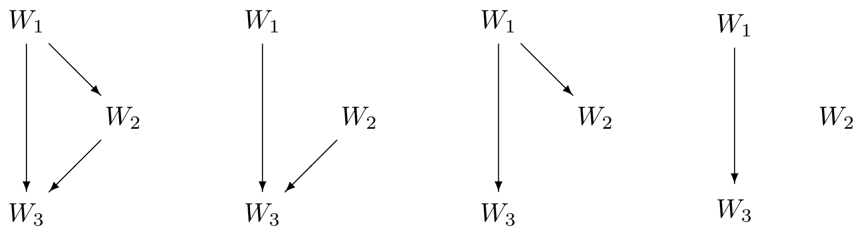
Remark 1.7 *In case of polarized varieties with low* Δ*-genera, there is an explicit classification theory such as [3]. However, it is desperate to classify polarized schemes with low* Δ*-genera. In fact, for a given closed subvariety* $X \subseteq \mathbb{P}^N$*, adding enough many closed points or lines, we can make a polarized scheme with low* Δ*-genus freely. Contrary to the facts above, if we once fix a closed subvariety* $V \subseteq \mathbb{P}^N$ *with low* Δ*-genus, it is nearly practical in our experience to classify the homological shells of* V*.*

Remark 1.8 (monotonously Δ-decreasing) *Let* P *and* $V \subseteq P$ *be the same as in the Conjecture (1.3) above and assume that the scheme* V *is arithmetically* D_2*. Suppose we have two homological shells* W *and* Z *of* V *with* $W \subseteq Z$*. Then, obviously the scheme* Z *is a homological shell of* W *and the scheme* W *is arithmetically* D_2*. If the conjecture (1.3.2) is true, then we have* $\Delta(W.O_W(1)) \geq \Delta(Z, O_Z(1))$*. This phenomenon is called as* monotonously Δ-decreasing *on the homological shells of* V*. To find evidence for the conjecture (1.3.2) through the classification of homological shells of a canonical curve, we have only to check the inequality on* Δ*-genera among its homological shells with inclusion relation.*

Definition 1.9 (maximal inclusion diagram) *Once an embedding of a projective variety* $X \subseteq \mathbb{P}^N = P$ *is given, we draw a diagram what we call* a maximal inclusion diagram *in the following rule in which we put* X *at the top and* P *at the bottom. As the rule, if there are two homological shells* W *and* W' *of* X *and there is a possibility of the existence of an inclusion* $W \subseteq W'$ *from the view point of graded Betti numbers, putting* W *at the up side and* W' *at the down side, we draw down an arrow* $W \longrightarrow W'$ *in the diagram. To explain more precisely, let us denote the i-th Betti number in degree* j *of* W *and of* W' *by* $\beta_{i,j}$ *and by* $\beta'_{i,j}$*, respectively. In other word,* $\beta_{i,j} = \dim_{\mathbb{C}} Tor_i^S(R_W, S/S_+)_{(j)}$ *and so on. Now we assume that there is an inclusion* $W \subseteq W'$*. Then, automatically, the scheme* W' *is a homological shell of* W *and therefore* $\beta_{i,j} \geq \beta'_{i,j}$ *for all* i *and* j*. Thus, by comparing the graded Betti numbers, namely if for all* i *and* j*,* $\beta_{i,j} \geq \beta'_{i,j}$ *hold, then we draw an arrow :* $W \to W'$ *in the diagram.*

Remark 1.10 *We have changed the name of the diagram as "maximal inclusion diagram" from "maximum inclusion diagram" in our previous paper [23]. Here we use the word "maximal" to show the set of inclusions is maximal, namely some of arrows for inclusions might be missing in the real circumstances.*

However, for simplicity, we omit to draw arrows for compositions of inclusions. For example, when we draw $W_1 \to W_2 \to W_3$ *in the maximal inclusion diagram, if their composed inclusion exists, then it may happen one of the* 4 *diagrams below in the real.*



Example 1.11 *To get used to a maximal inclusion diagram, let us review our classification on the good homological shells of a canonical curve* $X \subseteq \mathbb{P}^5 = P$ *which is of* $g = 6$ *and of generic case (non-trigonal and non-plane-quintic) handled in our previous paper [23]. The diagram is almost the same as the one in the last page of [23] except adding the term* (Q', Δ) *with duplication and confirming that the scheme* Y *is a variety. To see that the arithmetically Cohen-Macaulay scheme* Y *of dimension* 2 *is a variety, we compare the curve* X *and the scheme* Y*. The numbers* $a_1(X)$ *and* $a_1(Y)$*, namely the number of quadric equations of* X *and that of* Y*, respectively have a relation :* $a_1(X) = a_1(Y) + 1$*. The curve* X *and the scheme* Y *have no equation in degree* ≥ 3 *(as the members of minimal generators of their homogeneous ideals). This means that there is a quadric hypersurface* Q' *which satisfies* $X = Y \cap Q'$ *and* $Y \not\subseteq Q'$*.*

Then the quadric equation of Q' is a non-zero divisor for the homogeneous coordinate ring R_Y, otherwise the hypersurface Q' goes through an associated point of the scheme Y and $\dim(Y \cap Q') = 2$. Since the ring R_Y is arithmetically Cohen-Macaulay and the ring R_X is a domain, we see that the ring R_Y is also a domain which implies that the scheme Y is also a variety. The maximal inclusion diagram of the good homological shells of X is given as follows. Each mark on the right shoulder of each term has a meaning : $*$="classification accomplished", v="confirmed to be a variety", e="an example exists", respectively. Moreover, $(e_1, e_2, e_3) = (2, 1, 0)$ or $(1, 1, 1)$.

(#-1)

$$\begin{array}{ccc}
(X = Y \cap Q', \Delta = 5)^{[*,v]} & & \\
\downarrow & \searrow & \\
(Z, \Delta = 2)^{[e]} & & (Y, \Delta = 1)^{[v,e]} \\
\downarrow & \searrow & \downarrow \\
(Q \cap Q', \Delta = 1)^{[*,v]} & & (S(e_1, e_2, e_3), \Delta = 0)^{[*,v]} \\
\downarrow & \searrow & \downarrow \\
(Q', \Delta = 0)^{[*,v]} & & (Q, \Delta = 0)^{[*,v]} \\
& \searrow & \downarrow \\
& & (P, \Delta = 0)^{[*,v]}.
\end{array}$$

Problem 1.12 *From the diagram (#-1), a problem comes into our mind. For a projective subvariety $Y \subseteq \mathbb{P}^N = P$, we set the family of (good) homological shells of Y: $HSF(Y) := \{\text{(good) homological shell of } Y\} \subseteq \mathfrak{P}(P)$, where $\mathfrak{P}(P)$ denotes the power set of P. Let us assume that a projective subvariety $X \subseteq \mathbb{P}^N = P$ is represented as the transverse intersection of a projective subvariety $V \subseteq P$ with a hypersurface $D \subseteq P$, then does the following equality hold as the subsets of $\mathfrak{P}(P)$?*

$$HSF(X) \stackrel{?}{=} HSF(V) \cup \{W \cap D \mid W \in HSF(V)\}$$

Remark 1.13 *In the sequel, we often apply the argument of taking a main (irreducible) component W_0 of an arithmetically Cohen-Macaulay scheme $W \subseteq \mathbb{P}^N$. Here we should make a remark that the component W_0 is not arithmetically Cohen-Macaulay nor linearly normal in general. Let us use again the example of Remark 1.5. Take a cubic homogeneous polynomial $F := (Z_1^3 - Z_0^2 Z_2) + (Z_2^3 - Z_1 Z_3^2)$ and set $W := \{Z_0 Z_3 - Z_1 Z_2 = F = 0\}$ to be a $(2,3)$-complete intersection. Then the scheme $W \subseteq \mathbb{P}^3$ is arithmetically Cohen-Macaulay and has an irreducible decomposition $X \cup \ell_1 \cup \ell_2$, where X denotes the rational quartic curve in Remark 1.5 and ℓ_i does a line. If we take a main (irreducible) component X as W_0, then W_0 is not linearly normal, and therefore is not arithmetically Cohen-Macaulay.*

§2 Main Results.

Let us summarize our results in this article.

In the Petri's Analysis (cf. [8], [9]), there are 3 classes in the canonical curves of genus 6 : (i) plane quintic case ; (ii) trigonal case ; (iii) generic case (non-trigonal and non-plane-quintic). Our previous paper [23] handled the cases (iii) only. Now we study the case (i) in this article.

From the view point of Remark 1.8, we classify roughly homological shells of the canonical curve in this case and check their Δ-genera. Next theorem gives an evidence for the conjecture (1.3.2).

Theorem 2.1 *Let $X \subseteq \mathbb{P}^5(\mathbb{C}) = P$ be a canonical curve with $g = 6$ and of plane quintic type. Then, monotonously Δ-decreasing on* good *homological shells of X holds. Namely, take any two* good *homological shells Y and Z of the curve X with $Y \subseteq Z$. Then we always have :*

$$\Delta(Y, O_Y(1)) \geq \Delta(Z, O_Z(1))$$

.

Our rough classification on homological shells of a canonical curve of genus 6 and of plane quintic type is given as follows. We should make a remark that our classification on 2-dimensional homological shells is not finished yet, which is excluded here and will be handled in a forthcoming paper.

Theorem 2.2 *Let $X \subseteq \mathbb{P}^5(\mathbb{C}) = P$ be a canonical curve with $g = 6$ and of plane quintic type, and a* scheme *W be a homological shell of X.*

(2.2.1) *If $\dim W \neq 3$, then the homological shell is good and arithmetically Cohen-Macaulay.*

(2.2.2) *If $\dim W \neq 2$, the* good *homological shell W is irreducible and reduced.*

(2.2.3) *If $\dim W = 1$, then $W = X$.*

(2.2.4) *If $\dim W = 2$, then $\Delta(W, O_W(1)) = 0,\ 2,\ 3$. If moreover $\Delta(W, O_W(1)) = 0$, then it coincides with the Veronese surface $(\mathbb{P}^2, O_{\mathbb{P}^2}(2))$ defined by all the quadric equations of the curve X, which is simply denoted by V_e in the maximal inclusion diagram below. The data for W with $\Delta(W, O_W(1)) = 2,\ 3$ are described in Table 1.*

(2.2.5) *If $\dim W = 3$ and the homological shell W is good, then $\Delta(W, O_W(1)) = 0,\ 2$. If moreover $\Delta(W, O_W(1)) = 0$, it coincides with the the rational scroll $S(2, 1, 0)$, which is also a cone of $\mathbb{F}_1$.*

(2.2.6) *If $\dim W = 4$, then W is a quadric hypersurface Q or a cubic hypersurface D which is a member of minimal generators of the homogeneous ideal $\mathbb{I}_X$ of the curve X.*

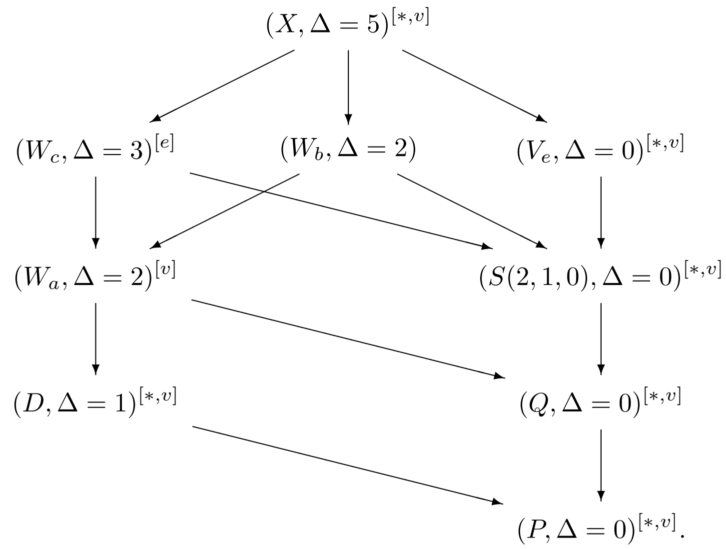
The data for homological shells remaining unknown are listed in Table 1, where a_1 and b_1 denote the number of equations in degree 2 and 3, respectively.

shells	*dim*	a_1	b_1	*Hilbert Poly.*	Δ
W_a	*3*	1	2	$5A_3 - 6A_2 + 2A_1 + 0A_0$	2
W_b	*2*	4	*?*	$6A_2 - 7A_1 + 2A_0$	2
W_c	*2*	3	*?*	$7A_2 - 9A_1 + 3A_0$	3

Table 1: The data of homological shells

The maximal inclusion diagram is given as follows. The symbol on the right shoulder of each term denotes $*$*="classification accomplished", v="confirmed to be a variety", e="an example exists", respectively.*

(#-2)



§3 Proof of the Results.

For a canonical curve $X \subseteq \mathbb{P}^5(\mathbb{C}) = P$ of genus $g = 6$ and of plane quintic type, an outline of the method of studying its homological shells is similar to that in [23]. However, the number of cases to be considered for this case is far larger than that for the generic case.

Let us start from studying graded Betti numbers of the homological shells of the curve X. As usual, we set $S = \mathbb{C}[Z_0, \cdots, Z_5]$, $S_+ = (Z_0, \cdots, Z_5)S$, and W to be a *good* homological shell of X. By Remark 1.2 (ii), the scheme W is also arithmetically Cohen-Macaulay.

As is well-known(cf. [10], [11]), the minimal graded S-free resolution $\mathbb{F}_{X,\bullet}$ of the homogeneous coordinate ring R_X is known to be :

(#-3)

$$0 \longleftarrow R_X \longleftarrow S \longleftarrow S(-2)^6 \oplus S(-3)^3 \longleftarrow S(-3)^8 \oplus S(-4)^8$$

$$\longleftarrow S(-4)^3 \oplus S(-5)^6 \longleftarrow S(-7)^1 \longleftarrow 0.$$

From the conditions of homological shells, each term of the minimal S-free resolution of the homogeneous coordinate ring R_W is a direct factor of the corresponding term in the resolution for R_X. Thus we have a minimal graded S-free resolution $\mathbb{F}_{W,\bullet}$:

(#-4)

$$0 \longleftarrow R_W \longleftarrow S \longleftarrow S(-2)^{a_1} \oplus S(-3)^{b_1} \longleftarrow S(-3)^{a_2} \oplus S(-4)^{b_2}$$

$$\longleftarrow S(-4)^{a_3} \oplus S(-5)^{b_3} \longleftarrow S(-7)^{b_4} \longleftarrow 0.$$

Of course, we have $0 \leq a_1 \leq 6$, $0 \leq b_1 \leq 3$, $0 \leq a_2 \leq 8$, $0 \leq b_2 \leq 8$, $0 \leq a_3 \leq 3$, $0 \leq b_3 \leq 6$, and $0 \leq b_4 \leq 1$ as the graded Betti numbers (cf. $a_1 = \beta_{1,2}$, $b_2 = \beta_{2,4}$ etc., where $\beta_{i,j}$ stands for the i-th Betti number in degree j). Now we consider the Hilbert polynomial $A_W(m)$ of the scheme W :

$$A_W(m) = \sum_{k=0}^{5} p_k(W) A_k(m) \qquad A_k(x) := \begin{pmatrix} x+k \\ k \end{pmatrix}.$$

By applying Lemma 2.6 in [15], we can write down the Hilbert polynomial $A_W(m)$ of W which is described by these graded Betti numbers.

$$\begin{cases} p_5 = p_5(W) = 1 - a_1 - b_1 + a_2 + b_2 - a_3 - b_3 + b_4 \\ p_4 = p_4(W) = 2a_1 + 3b_1 - 3a_2 - 4b_2 + 4a_3 + 5b_3 - 7b_4 \\ p_3 = p_3(W) = -a_1 - 3b_1 + 3a_2 + 6b_2 - 6a_3 - 10b_3 + 21b_4 \\ p_2 = p_2(W) = +b_1 - a_2 - 4b_2 + 4a_3 + 10b_3 - 35b_4 \\ p_1 = p_1(W) = b_2 - a_3 - 5b_3 + 35b_4 \\ p_0 = p_0(W) = b_3 - 21b_4 \end{cases}$$

Now we set $w = \dim W$, $d = \deg W$, and $\Delta = \Delta(W, O_W(1))$. Then $w = 1, 2, \ldots, 5$.

• Assume that $w = 1$. Then $p_5 = p_4 = p_3 = p_2 = 0$ and $p_1 > 0$. Then we see that $d = p_1 = 10$ and $hd_S(R_W) \leq hd_S(R_X)$, which implies $arith.depth(R_W) \geq arith.depth(R_W) = 2$, namely the scheme W is arithmetically Cohen-Macaulay. Applying Lemma 3.6 in [20], we get $W = X$ and $\Delta = 5$.

• Suppose that $w = 5, 4$. If $w = 5$, obviously $W = P$. If $w = 4$, then $a_1 + b_1 \geq 1$. Since the curve X is irreducible and reduced, all the minimal generators of $\mathbb{I}_X$ is irreducible, which implies $a_1 + b_1 = 1$, namely the scheme W is a divisor of P. By the classical works [8] or [9], we see that the scheme W is a hypersurface of $d = 2$ or $d = 3$.

• Let us consider the case $w = 3$, namely $p_5 = p_4 = 0$ and $p_3 > 0$, which implies $d = p_3$.

Then we have 75 cases of $(a_1, b_1, a_2, b_2, a_3, b_3, b_4, d)$. For example, we have a case : $(a_1, b_1, a_2, b_2, a_3, b_3, b_4, d) = (3, 1, 1, 5, 1, 2, 0, 1)$. It might happen that the scheme W is a scheme theoretic union of 3-plane and a scheme of dim ≤ 2 which includes the canonical curve X. It is not so easy to exclude this case by analyzing its scheme structure and syzygies. Thus we assume that the homological shell W is good, namely $a_3 = b_3 = b_4 = 0$.

By Remark 1.2 (ii), the scheme W is arithmetically Cohen-Macaulay, which implies the surjectivity of the natural map $H^0(P, O_P(1)) \to H^0(W, O_W(1))$. On the other hand, since $X \subseteq W$, the scheme W is also linearly non-degenerate, and therefore $H^0(P, O_P(1)) \cong H^0(W, O_W(1))$, namely $h^0(W, O_W(1)) = 6$ and $\Delta = d + 3 - 6 = d - 3$.

Hence, 10 cases of (a_1, b_1, a_2, b_2) remain as in the following table.

Case No.	(a_1, b_1, a_2, b_2)	d	Δ	Existence
(1)	$(1, 2, 0, 2)$	5	2	?
(2)	$(1, 3, 1, 2)$	5	2	No
(3)	$(2, 0, 0, 1)$	4	1	No
(4)	$(2, 1, 1, 1)$	4	1	No
(5)	$(2, 2, 2, 1)$	4	1	No
(6)	$(2, 3, 3, 1)$	4	1	No
(7)	$(3, 0, 2, 0)$	3	0	Yes
(8)	$(3, 1, 3, 0)$	3	0	No
(9)	$(3, 2, 4, 0)$	3	0	No
(10)	$(3, 3, 5, 0)$	3	0	No

Table 2: The graded Betti Numbers, the degree and Δ

Depending on $d = 3, 4, 5$, let us consider these 10 cases.

•• Now we assume $d = 3$. Take a main component W_0 including the curve X. If the scheme W is reducible or non-reduced, then the scheme $(W_0)_{red}$ with the reduced structure has $\deg((W_0)_{red}) \leq 2$, which implies the scheme $(W_0)_{red}$ is linearly degenerate. It contradicts the fact $X \subseteq (W_0)_{red}$. Hence we see that the scheme W is a variety. Then the variety W is a variety of minimal degree and whose homogeneous coordinate ring R_W has a 2-linear resolution, which excludes the 3 cases (8)-(10). Only the case (7) remains. Take the quadric hull of the curve X, namely the Veronese surface V_e as in [8] or [9] since we assume that the curve X is of plane quintic type. The variety W is defined by 3 quadric equations of X, we see that $X \subseteq V_e \subseteq W$. Hence the variety W is a homological shell of V_e. Applying the result of [16], we see that $W \cong S(2, 1, 0) = \mathbb{P}(O(2, 1, 0))$, namely the one point codal variety of V_e or equivalently the cone of the image of inner projection of V_e which is the same as the cone of the one point blow up $\mathbb{F}_1$ of $\mathbb{P}^2$. Then the existence of this case is obvious.

•• Next we consider the case $d = 4$. By Table 2, we see that $a_1 = 2$. Let us take two linearly independent quadric equations f_1, f_2 of W. Both the two equations f_1 and f_2 are irreducible because they form a part of minimal generators of the homogeneous prime ideal $\mathbb{I}_X$. Set the scheme W^* to be the $(2, 2)$-complete intersection defined by $f_1 = f_2 = 0$. Then $X \subseteq W \subseteq W^*$. These two scheme W and W^* are arithmetically Cohen-Macaulay , of dimension 3 and $\deg W = \deg W^* = 4$, which implies $W = W^*$ by Lemma 3.6 of [20]. Then the 3 cases (4)-(6) are excluded and only the case (3) remains. Now, by the similar argument in the case $d = 3$ above, we see that $X \subseteq V_e \subseteq W$ where V_e denotes the Veronese surface defined by the quadric hull of the curve X. Then, it implies that the scheme W is a homological shell of V_e. Hence we have an injective map $\mathbb{C} \cong Tor_2^S(R_W, S/S_+)_{(4)} \to Tor_2^S(R_{Ve}, S/S_+)_{(4)} = 0$, which is a contradiction.

•• Let us consider the remaining case $d = 5$, which means the case (1) or (2). First we study the case (2). Take a minimal graded S-free resolution of R_W:

$$0 \longleftarrow R_W \longleftarrow S \xleftarrow{\varphi_1} S(-2)^1 \oplus S(-3)^3 \xleftarrow{\varphi_2} S(-3)^1 \oplus S(-4)^2 \longleftarrow 0.$$

By the minimality of the resolution, the map φ_1 on the component $S(-2)^1$ is represented by a non-zero homogeneous quadric equation Q, and the map φ_2 on the component $S(-3)^1$ is represented by a non-zero vector $[L, 0, 0, 0]$ where L denpotes a linear homogeneous polynomial. Then, the fact $\varphi_1 \circ \varphi_2 = 0$ implies $Q \cdot L = 0$, which is a contradiction. Hence the case (2) in Table 2 is excluded. Thus we consider the case (1) only in the sequel.

Take a main component W_0 including the curve X and set the scheme $(W_0)_{red}$ to be the reduced scheme. By the similar argument in the case $d = 3$, the scheme $(W_0)_{red}$ is linearly non-degenerate, which means $\deg(W_0)_{red} \geq 3$. Let η be the generic point of the scheme W_0 and set $s := length O_{W_0,\eta}$. Since $(W_0)_{red} \subseteq W_0 \subseteq W$, by using Lemma 3.7 in [20], we see that $3 \leq 3 \cdot s \leq 5$, which shows $s = 1$, namely $(W_0)_{red} = W_0$, or equivalently the scheme W_0 is reduced and a variety. Set $d_0 := \deg W_0$, then $d_0 = 3, 4, 5$.

Now we suppose $d_0 = 3$. Then the variety W_0 is of minimal degree and has 2-linear resolution. Hence $Tor_2^S(R_{W_0}, S/S_+)_{(4)} = 0$. By the inclusions $X \subseteq W_0 \subseteq W$, the scheme W is also a homological shell of W_0, which induces an injection $Tor_2^S(R_W, S/S_+)_{(4)} \to Tor_2^S(R_{W_0}, S/S_+)_{(4)}$. This is absurd because $Tor_2^S(R_W, S/S_+)_{(4)} \cong \mathbb{C}^2$ which is brought by $b_2 = 2$ in the cases (1) of Table 2.

Next we consider the case $d_0 = 4$. The inclusion $X \subseteq W_0$ shows $h^0(W_0, O_{W_0}(1)) \geq 6$. By using the Fujita's inequality on Δ-genus (cf. (4.2)Theorem in [3]), we have

$$0 \leq \Delta(W_0, O_{W_0}(1)) = 4 + 3 - h^0(W_0, O_{W_0}(1)) \leq 7 - 6 = 1,$$

which implies $h^0(W_0, O_{W_0}(1)) = 6, 7$. Let us suppose $h^0(W_0, O_{W_0}(1)) = 6$, namely the variety W_0 is linearly normal. Then, by [12], the scheme W_0 is a $(2, 2)$ complete intersection, which implies $Tor_1^S(R_{W_0}, S/S_+)_{(3)} = 0$. From the inclusions $X \subseteq W_0 \subseteq W$, the scheme W is a homological shell of W_0, which shows that $Tor_1^S(R_W, S/S_+)_{(3)} = 0$ and $b_1 = 0$. By comparing with the case (1) in Table 2, we see that this is absurd. Now we study the case $h^0(W_0, O_{W_0}(1)) = 7$. Then, there exist a linearly non-degenerate variety $\widetilde{W_0} \subseteq \mathbb{P}^6 = \widetilde{P}$ of dimension 3, a curve $\widetilde{X} \subseteq \widetilde{W_0}$, and a linear projection $\pi : \widetilde{P} = \mathbb{P}^6 \cdots \to \mathbb{P}^5 = P$ which sends isomorphically the variety $\widetilde{W_0}$ and the curve $\widetilde{X}$ to the variety W_0 and to the curve X, respectively. Since $\pi^* O_{W_0}(1) \cong O_{\widetilde{W_0}}(1)$, $\deg \widetilde{W_0} = 4$, $h^0(\widetilde{W_0}, O_{\widetilde{W_0}}(1)) = 7$, $\Delta(\widetilde{W_0}, O_{\widetilde{W_0}}(1)) = 0$, the variety $\widetilde{W_0}$ is arithmetically Cohen-Macaulay. Similarly, $\pi^* O_X(1) \cong O_{\widetilde{X}}(1)$ and $h^0(\widetilde{X}, O_{\widetilde{X}}(1)) = h^0(X, O_X(1)) = h^0(X, \omega_X) = 6$. Hence the curve $\widetilde{X}$ is linearly degenerate in $\widetilde{P}$, namely there is a 5-plane $\widetilde{H} \subseteq \widetilde{P}$ which contains the curve $\widetilde{X}$. This brings the inclusions $\widetilde{X} \subseteq \widetilde{W_0} \cap \widetilde{H} \subseteq \widetilde{H} \cong \mathbb{P}^5$. The scheme $\widetilde{V} := \widetilde{W_0} \cap \widetilde{H}$ is arithmetically Cohen-Macaulay of dimension 2 with degree 4. If the scheme $\widetilde{V}$ is reducible or non-reduced, by taking a main component including the curve $\widetilde{X}$ of $\widetilde{V}$, the similar argument above gives a contradiction. Now we see that the surface $\widetilde{V}$ is of minimal degree and defined by quadric equations. Recalling the fact that the embedding $\widetilde{X} \subseteq \widetilde{H} \cong \mathbb{P}^5$ is the canonical embedding, we see that the surface $\widetilde{V}$ is a Veronese surface defined by all the quadric equations of $\widetilde{X}$ in $\widetilde{H} \cong \mathbb{P}^5$. Because the curve $X \subseteq P$ is linearly non-degenerate, the linear projection sends the 5-plane $\widetilde{H}$ isomorphically to the 5-th projective space P. This shows that the linear projection π sends the Veronese surface $\widetilde{V}$ isomorphically to its image $\pi(\widetilde{V})$. Namely, the inclusions $\widetilde{X} \subseteq \widetilde{V} \subseteq \widetilde{W_0}$ are sent to the inclusions $X \subseteq \pi(\widetilde{V}) \subseteq W_0$ isomorphically by the linear projections π, respectively. By the reason that the surface $\pi(\widetilde{V})$ is also a Veronese surface, we get $\pi(\widetilde{V}) = V_e$, which implies the inclusions $X \subseteq V_e \subseteq W_0 \subseteq W$. Hence we get the scheme W is a homological shell of the Veronese surface V_e. Since $Tor_1^S(R_{Ve}, S/S_+)_{(3)} = 0$, we see $b_1 = 0$, which contradicts the case (1) in Table 2.

Thus we show that $d_0 = 5$, namely the scheme W is an arithmetically Cohen-Macaulay variety of dimension 3 with $d = 5$. From the graded Betti numbers of the case (1) in Table 2, we obtain that $p_3 = d = 5$, $p_2 = -6$, $p_1 = 2$, $p_0 = 0$, which implies the sectional genus $g(W, O_W(1)) = 2$.

Once we obtain a linearly non-degenerate arithmetically Cohen-Macaulay variety of dimension 3 with degree 5 and the sectional genus $g(W, O_W(1)) = 2$ in $\mathbb{P}^5$, we can calculate conversely the graded Betti numbers and confirm that they coincides with the ones in the case (1) of Table 2. The strategy of calculating the graded Betti numbers is the same as the one in the proof of Theorem 2.4 in [21]. In fact, by cutting the variety W with generic 3-plane L, we obtain an arithmetically Cohen-Macaulay integral scheme Y of dimension 1 in $L \cong \mathbb{P}^3$, which has the arithmetic genus $p_a(Y) = 2$. Then, using Hilbert polynomial of Y, we can recover this minimal graded S_L-free resolution of the ring R_Y, where S_L denotes the polynomial ring of $L \cong \mathbb{P}^3$. Thus we obtain the graded Betti numbers of W as in the case (1).

However, at this stage, we can not show the existence of the homological shell W of this type yet.

• Now we assume $w = 2$, namely $p_5 = p_4 = p_3 = 0$ and $p_2 > 0$, which implies $d = p_2$. Without assuming goodness of the homological shell, calculation of the graded Betti numbers under the assumption above shows that we have 42 cases of $(a_1, b_1, a_2, b_2, a_3, b_3, b_4)$, where any case satisfies $b_4 = 0$. Namely the homological shell becomes automatically good, and therefore arithmetically Cohen-Macaulay. Since the scheme W is linearly non-degenerate and arithmetically Cohen-Macaulay, we get $h^0(W, O_W(1)) = 6$. From the calculation of the graded Betti numbers, we see that $d = 4, 5, 6, 7$ and $\Delta = 0, 1, 2, 3$, respectively.

•• If $d = 4$, then $a_1 = 6$, which means that the scheme W is defined all the quadric equations of the curve X since $\dim Tor_1^S(R_X, S/S_+)_{(2)} = 6$. This shows that the scheme W coincides with the Veronese surface V_e. As we already showed in [16], [18], and [22], any homological shell of the variety of $\Delta = 0$ is also a variety of $\Delta = 0$. Hence, the Veronese surface is not contained by any homological shell of the curve X with dimension 3 and $\Delta = 2$.

•• Let us study roughly the case $d = 5, 6, 7$. We have already showed that for $w = 3$, $\Delta = 0, 2$. Hence, to show the monotonously Δ-decreasing, for $w = 2$, we have only to exclude the case $\Delta = 1$, namely $d = 5$. Thus, the following lemma is a key for our proof.

Key Lemma 3.1 *Let $X \subseteq \mathbb{P}^5(\mathbb{C}) = P$ be a canonical curve with $g = 6$ and of plane quintic type. Assume that $W \subseteq P$ is a homological shell of X and $\dim W = 2$. Then, $d = \deg W \neq 5$.*

Proof. With the additional assumption $d = 5$, calculation of the graded Betti numbers under the assumption : $p_5 = p_4 = p_3 = 0$ brings that we have 16 cases of $(a_1, b_1, a_2, b_2, a_3, b_3, b_4)$, where any case satisfies $a_1 = 5$.

Let us take a main component W_0 of W which contains the curve X. Then we have that $X \subseteq (W_0)_{red} \subseteq W_0 \subseteq W$ and the variety $(W_0)_{red}$ is linearly non-degenerate. This shows that $5 \geq \deg(W_0)_{red} \geq 4$. As in the argument above, by considering the length of the local ring of W_0 at its generic point, we see that $W_0 = (W_0)_{red}$.

If $\deg W_0 = 4$, then the surface W_0 is a variety of minimal degree and defined by quadric equations of the curve X. Then $V_e \subseteq W_0$, where V_e denotes the Veronese surface including the curve X. Comparing their dimensions, we get $V_e = W_0$. Namely $X \subseteq V_e \subseteq W$, which implies that the scheme W is also a homological shell of V_e.

Applying the result of [16], [18], and [22], we see that the scheme W is a variety of $\Delta = 0$, which contradicts the assumption : $d = 5$, namely $\Delta = 1$.

Thus we have $\deg W_0 = 5$, namely $W = W_0$. In other word, the scheme W is an arithmetically Cohen-Macaulay variety of $d = 5$. By using $a_1 = 5$ and $\dim Tor_1^S(R_X, S/S_+)_{(2)} = 6$, we can find a quadric hypersurface Q defined by a homogeneous polynomial f of degree 2 which satisfies $X \subseteq Q$ and $W \not\subseteq Q$. Then the scheme $W \cap Q$ is arithmetically Cohen-Macaulay, contains the curve X, and has $\deg(W \cap Q) = 5 \cdot 2 = 10 = \deg X$. Hence we have $X = W \cap Q$, which induces an exact sequence:

$$0 \longleftarrow R_X \longleftarrow R_W \xleftarrow{f} R_W(-2) \longleftarrow 0.$$

Applying the minimal graded S-free resolution $\mathbb{F}_{W\bullet}$ in (#-4) to R_W and $R_W(-2)$, respectively in the sequence above, we obtain a double complex whose total complex gives a minimal grade S-free resolution of R_X:

(#-5)

$$\begin{array}{ccccccccc} S(-2)^1 & \longleftarrow & S(-4)^5 \oplus S(-5)^{b_1} & \longleftarrow & S(-5)^{a_2} \oplus S(-6)^{b_2} & \longleftarrow & S(-6)^{a_3} \oplus S(-7)^{b_3} & \longleftarrow & \\ f\downarrow & & f\downarrow & & f\downarrow & & f\downarrow & & \\ S & \longleftarrow & S(-2)^5 \oplus S(-3)^{b_1} & \longleftarrow & S(-3)^{a_2} \oplus S(-4)^{b_2} & \longleftarrow & S(-4)^{a_3} \oplus S(-5)^{b_3} & \longleftarrow & . \end{array}$$

Recalling the fact that the minimal graded S-free resolutions are unique up to isomorphisms as complexes, let us compare $\mathbb{F}_{X,\bullet}$ in (#-3) with the total complex of (#-5). First we check the term $\mathbb{F}_{X,1}$, which shows $S(-2)^1 \oplus (S(-2)^5 \oplus S(-3)^{b_1}) \cong S(-2)^6 \oplus S(-3)^3$ and implies $b_1 = 3$. Next we see the term $\mathbb{F}_{X,2}$, which shows $(S(-3)^{a_2} \oplus S(-4)^{b_2}) \oplus (S(-4)^5 \oplus S(-5)^{b_1}) \cong S(-3)^8 \oplus S(-4)^8$ and gives $b_1 = 0$. This is a contradiction. ∎

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