Homological shells of a canonical curve of genus 5 or 6 (I)

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Abstract

We continue to classify the homological shells of a canonical curve with genus $g = 5$ or $g = 6$. In the case $g = 5$, we solve affirmatively the remaining problem on the existence of a homological shell surface with degree 5 for any trigonal canonical curve and finish our classification in this case. For the case $g = 6$, assuming that the curve is generic (i.e. non-trigonal and non-plane-quintic), we investigate mainly good homological shells of dimension 3. We also show the inequality on Δ-genera of homological shells coming from good homological shells of a canonical curve with $g = 6$, which is predicted by our Δ-genus inequality conjecture in [13].
Keywords: (good) homological shell, pregeometric shell, canonical curve, trigonal curve, plane quintic curve, genus 5 or 6

§0 Introduction.

In [13] and [17], we presented several problems based on our faith that there must exist a "geometry" of projective embeddings which reflects the intrinsic or internal geometry of projective varieties. To realize our virtual geometry on projective embeddings, we pay a special attention to intermediate ambient schemes which satisfy certain good conditions from the view point of syzygies for the given embedded variety. Those intermediate ambient schemes are called as homological shells (previously called as "pregeometric shells"), whose precise definition is given in Definition 1.1, and was first introduced in [12]. Among the problems in [13] and [17], Conjecture (1.3) including Δ-genus inequality conjecture (1.3.2) is the most interesting and fundamental one. In a series of our articles [14], [15], [16], [18], [19], [20], we found several evidences for the conjectures by classifying homological shells of a given embedded projective variety.

This article is also a part of the series mentioned above. Here, we take a canonical curve of genus $g = 5$ or $g = 6$(with an assumption "generic") as the embedded projective variety and study its homological shells.

In the case $g = 5$, for a generic, namely a non-trigonal canonical curve, all its homological shells are classified at a glance, which shows easily that the conjectures (1.3.1) and (1.3.2) hold in this case. On the other hand, for a trigonal canonical curve, we have to consider precisely its homological shells and find the conjectures (1.3.1) and (1.3.2) holding also in this case. However, the key part of classification of its homological shells, which is the case that the homological shells have dimension 2, was not completed within [18]. There we showed that a homological shell of the curve with dimension 2 is irreducible,

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Typeset by $\mathcal{AMS}$-LaTeX with P.Burchard's diagram package

reduced, arithmetically Cohen-Macaulay, and has the degree 3 or 5 if it exists. It is obvious to see that any trigonal canonical curve of $g = 5$ has a unique smooth homological shell surface of degree 3. On the other hand, it was not so easy for us to see that any trigonal canonical curve of $g = 5$ has a homological shell surface of degree 5. In the subsequent article [19], for a special trigonal canonical curve of $g = 5$, we construct a smooth homological shell surface of degree 5 by applying geometric method, and a singular homological shell surface of degree 5 by applying algebraic method, respectively.

In this article, by using Brill-Noether theory, we show that every trigonal canonical curve of $g = 5$ has a homological shell surface of degree 5. In the case $g = 6$, we study a generic canonical curve, namely a non-trigonal and non-plane-quintic canonical curve, which is defined by quadric equations (cf. Petri's analysis [8], [9]). Even in this case, there are still too many subcases to handle, we restrict our selves to classify "good" homological shells. For good homological shells of a generic canonical curve of genus $g = 6$, we can confirm that Δ-genus inequality conjecture (1.3.2) holds. As a help to prove the conjecture only in this case, we also classify mainly its homological shells of dimension 3. We also add newly Conjecture 1.8 arising from this work.

Also in this article, we use successively the notation and conventions in [5] and in [18] without mention.

§1 Preliminaries.

Let us recall our key concept for studying the geometric structures of projective embeddings. The concept "homological shell" was introduced first in [12]. We can find many good actual examples of this concept in a number of classical works in Complex Projective Geometry such as [8], [9], [11], [3] and so on.

Definition 1.1 (shells and cores) *Take a polynomial ring $S := \mathbb{C}[Z_0, \dots Z_N]$ of $(N+1)$-variables over the complex number field $\mathbb{C}$ with the usual grading, and its maximal homogeneous ideal $S_+ := (Z_0, \dots Z_N)S$. Let V and W be closed sub<u>schemes</u> of $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ which satisfy $V \subseteq W$ (namely the inclusion of the defining ideal sheaves: $I_V \supseteq I_W$ in the structure sheaf O_P of P; In this case, the subscheme W is called simply an* intermediate ambient scheme *of V).*

(1.1.1) *If the natural map:*

$$\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ (abbr. "global Tor injectivity condition"*), we say that W is a* homological shell (*abbr.* H-shell) *of V and that V is a* homological core (*abbr.* H-core) *of W, where $R_W := S/\mathbb{I}_W$ and $R_V := S/\mathbb{I}_V$ denote the homogeneous coordinate rings of W and of V, respectively, and $\mathbb{I}_W := \oplus_m H^0(P, I_W(m))$, $\mathbb{I}_V := \oplus_m H^0(P, I_V(m))$.*

(1.1.2) *Now we set $r_0 := \dim(W) - \dim(V)$. Assume that the scheme W is a homological shell of V and that for every integer $q \geq 0$, we have $Tor_{q-r_0}^S(R_W, S/S_+) = 0$ if $Tor_q^S(R_V, S/S_+) = 0$. Then we say that the scheme W is a* good *homological shell of V and the scheme V is a* good *homological core of W.*

For the subscheme V, the total space P and V itself are called as trivial *(good) homological shells.*

Remark 1.2 (i) *Homological shell defined in (1.1.1) above was called as* pregeometric shell *or* PG-shell *in our several previous works after we introduced this concept in [12].*

(ii) *The integer r_0 in (1.1.2) coincides with $codim(V, W)$ if the scheme W is irreducible. However, without assuming irreducibility of the scheme W, the integer r_0 does not coincide with $codim(V, W)$ in general.*

(iii) *The condition "good" in (1.1.2) is the same to say the inequality on the homological dimensions* $hd_S(R_W) \leq hd_S(R_V) - r_0$, *or equivalently* $arith.depth(W) \geq arith.depth(V) + r_0$ *by Auslander-Buchsbaum formula. Thus, supposing that the scheme* V *is arithmetically Cohen-Macaulay and that the scheme* W *is a homological shell of* V, *it is obvious that the scheme* W *is arithmetically Cohen-Macaulay if and only if the homological shell* W *is* good.

(iv) *Homological shell is not always good. For example, see the example given by Remark (1.4) in [20]. This example also shows that a closed subscheme which has an arithmetically Cohen-Macaulay homological core is not always arithmetically Cohen-Macaulay.*

Let us recall our two fundamental conjectures on homological shells from [13] and [14].

Conjecture 1.3 *Let* $P = \mathbb{P}^N(\mathbb{C})$ *be an N-th projective space with the tautological ample line bundle* $O_P(1) = O_P(H)$ *and* $V \subseteq W \subseteq P$ *its closed subschemes.*

(1.3.1) *Assume that the scheme* V *is a variety, namely reduced and irreducible and that the closed subscheme* W *is a homological shell of* V. *Then the subscheme* W *is also a variety.*

(1.3.2) [**Δ-genus inequality conjecture**] *Suppose that the subscheme* V *is arithmetically* D_2, *namely its arithmetic depth* ≥ 2. *If* W *is a homological shell of* V, *then the inequality:*

$$\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$$

holds on their Δ*-genera (cf. For a scheme* V, *its* Δ*-genus is defined by* $\Delta(V, O_V(1)) := dim(V) + deg(O_V(1)) - h^0(V, O_V(1))$ *; see* [3]).

Remark 1.4 *Previously, we assumed that both the schemes* V *and* W *are varieties in the conjecture (1.3.2) since* Δ*-genus is usually defined for a pair of a variety and an ample line bundle on it. However, the definition of* Δ*-genus is formally valid also for a pair of a scheme and an ample invertible sheaf on it. It is also convenient to generalize the statement of the conjecture (1.3.2) for closed schemes from the technical view point since the two conjectures can be handled independently. For additional information on these two conjectures, see §1 of [16].*

Remark 1.5 *Let* P *and* $V \subseteq P$ *be the same as in the Conjecture (1.3) above and assume that the scheme* V *is arithmetically* D_2. *Suppose we have two homological shells* W *and* Z *of* V *with* $W \subseteq Z$. *Then, obviously the scheme* Z *is a homological shell of* W *and the scheme* W *is arithmetically* D_2. *If the conjecture (1.3.2) is true, then we have* $\Delta(W.O_W(1)) \geq \Delta(Z, O_Z(1))$. *To find evidence for the conjecture (1.3.2) through the classification of homological shells of a canonical curve, we have only to check the inequality on* Δ*-genera among its homological shells with inclusion relation.*

Definition 1.6 *Let* P *and* $V \subseteq W \subseteq P$ *be the same as in the initial setting of Conjecture (1.3). The scheme* W *is called as a* layered *homological shell of* V *if there is a chain of homological shells* $\{Y_b\}_{b=0}^r$ *of* V : $W = Y_0 \subset Y_1 \subset \ldots \subset Y_r = P$ *and* $\dim Y_b = b + m$ $(b = 0, 1, \ldots, r)$, *where* $r = codim(W, P)$.

Remark 1.7 *Let* P *and* $V \subseteq P$ *be the same as in Conjecture (1.3). Suppose that the scheme* V *is linearly non-degenerate and that we obtain an intermediate ambient scheme* W *of* V *which is a variety of minimal degree, namely integral and* $\Delta(W, O_W(1)) = 0$. *Then the variety* W *is a layered homological shell of* V *as we saw in [19].*

Conjecture 1.8 *Let P and $V \subseteq P$ be the same as in Conjecture (1.3) above. If the scheme V is arithmetically Cohen-Macaulay, then any good homological shell W of V is a layered homological shell of V?*

Remark 1.9 *If the scheme V is a variety of $\Delta(V, O_V(1)) = 0$, then the statement of Conjecture 1.8 is affirmative by Remark 1.7 and [19].*

§2 Main Results.

Let us summarize our results in this article. The first one is to give an affirmative answer to our remaining problem in [18] and [19].

Theorem 2.1 *Let $X \subset P = \mathbb{P}^4(\mathbb{C})$ be a trigonal canonical curve of genus 5. Namely, taking a non hyperelliptic curve C of genus $g(C) = 5$ with a complete base point free linear system g_3^1 and its canonical embedding $\Phi_{|K_C|} : C \to P = \mathbb{P}^4(\mathbb{C})$, we set $X := \Phi_{|K_C|}(C)$. Then there always exists an integral homological shell surface Z of X with $\deg Z = 5$.*

An interesting application of the theorem above is the following criterion on trigonality.

Corollary 2.2 *Let $X \subset P = \mathbb{P}^4(\mathbb{C})$ be a canonical curve of genus 5. Then the canonical curve X is trigonal if and only if it has a homological shell surface Z of degree 5.*

We also add here the following result which has been essentially proved in [18]. This formulation is rather stronger than that of Main Theorem 2.1 in [18].

Theorem 2.3 *Let $X \subset P = \mathbb{P}^4(\mathbb{C})$ be a canonical curve of genus 5. Take any two homological shells Y and Z of the curve X with $Y \subseteq Z$. Then we always have : $\Delta(Y, O_Y(1)) \geq \Delta(Z, O_Z(1))$.*

Now we consider a canonical curve of genus $g = 6$. From the view point of Petri's Analysis (cf. [8], [9]), there are 3 classes in the canonical curves of genus 6 : (i) plane quintic case ; (ii) trigonal case ; (iii) generic case (non-trigonal and non-plane-quintic). Here we handle the cases (i) only. From the view point of Remark 1.5, we classify roughly homological shells of the canonical curve in this case and check their Δ-genera. Next theorem gives an evidence for the conjecture (1.3.2).

Theorem 2.4 *Let $X \subset P = \mathbb{P}^5(\mathbb{C})$ be a generic canonical curve of genus 6. Take any two* good *homological shells Y and Z of the curve X with $Y \subseteq Z$. Then we always have : $\Delta(Y, O_Y(1)) \geq \Delta(Z, O_Z(1))$.*

Our rough classification on homological shells of a generic canonical curve of genus 6 is given as follows. We should make a remark that our classification on 2-dimensional homological shells is not finished yet, which is excluded here and will be handled in a forthcoming paper.

Theorem 2.5 *Let $X \subset P = \mathbb{P}^5(\mathbb{C})$ be a generic canonical curve of genus 6 and W be a* good *homological shell of X. If $codim(W, P) \neq 3$, then the scheme W is irreducible, reduced and arithmetically Cohen-Macaulay.*

If $codim(W, P) = 4$, then $W = X$.

If $codim(W, P) = 1$, then W is a quadric hypersurface.

If $codim(W, P) = 2$, then $\Delta(W, O_W(1)) = 0$ or 1.

More explicit information on these homological shells will be given in the last section.

§3 Review on Classical Results.

Let us summarize the classical works on canonical curves mainly from [10], [11] without any proof. For later use, our notation is a little different from [11](e.g. the symbol $S(e_1, e_2, \ldots, e_m)$ is used not only for the type of a rational scroll but also for a rational scroll itself). When we consider a linear system, we always take the complete linear system otherwise mentioned particularly. Hence we do not distinguish the differences among line bundles, Cartier divisor classes, and linear systems.

Taking non negative integers $\{e_i\}_{i=1}^m$ with $e_1 \geq e_2 \geq \ldots \geq e_m \geq 0$, we set $B = \mathbb{P}^1(\mathbb{C})$, and the vector bundle $E = \oplus_{i=1}^m O_B(e_i)$ on the rational curve B, where the invertible sheaf $O_B(1)$ is the ample tautological line bundle of B. Set $d = e_1 + \cdots + e_m$, $N = d + m - 1$, a morphism φ to be the morphism from $U = P(E) = \mathbb{P}(E)$ to $P = \mathbb{P}^N(\mathbb{C})$ defined by the complete linear system $|O_{P(E)/B}(1)|$, a closed variety $W = S(e_1, e_2, \ldots, e_m) \subseteq P$ to be the image of φ, a morphism $\pi : P(E) \to B$ to be the structure morphism of the projective bundle, and assume $d \geq 2$. Since $deg(W) = d$, $\dim W = m$, the variety W is a variety of minimal degree, i.e. $\Delta(W, O_P(1) \otimes O_W) = 0$.

The morphism φ is an embedding if and only if $e_m > 0$. If $e_1 \geq e_2 \geq \ldots \geq e_k > 0 = e_{k+1} = \ldots = e_m$, then the variety $W = S(e_1, e_2, \ldots, e_m)$ is a $(d-k)$ multiple cone of the k-dimensional nonsingular projective subvariety $S(e_1, e_2, \ldots, e_k)$ and the morphism $\varphi : U \to W$ is a resolution of rational singularities. In this case, the singularities of $W = S(e_1, e_2, \ldots, e_m)$ coincides with the vertices, which is $(m-k-1)$-dimensional linear space.

The divisor classes $H = \varphi^* O_P(1)$, $R = \pi^* O_B(1)$ form a free basis of $Pic(P(E))$ as $\mathbb{Z}$-modules. In the Chow ring $A^\bullet(P(E))$, we have : $H^m = d, \quad H^{m-1} \cdot R = 1, \quad$ and $R^2 = 0$.

For integers $i > 0$, $a \in \mathbb{Z}$, $b \geq -1$, we have $R^i \varphi_* O_{P(E)}(aH + bR) = 0$, and set a coherent sheaf $O_W(aH + bR) = \varphi_* O_{P(E)}(aH + bR)$ including the case $b < -1$.

Let us take a section $\sigma_i \in H^0(P(E), O_{P(E)}(H - e_i R))$ which corresponds to an i-th direct factor $O_{P(E)}(e_i) \hookrightarrow E$. Taking a homogeneous coordinates $[s : t]$, we set $Z_{p.q} := s^{e_p - q} t^q \sigma_p$ $(p = 1, \ldots, m\,; q = 0, \ldots, e_p)$, and a $2 \times d$ matrix Φ to be :

$$\Phi = \begin{bmatrix} Z_{1,0} & \cdots & Z_{1,e_1-1} & Z_{2,0} & \cdots & Z_{2,e_2-1} & \cdots\cdots & Z_{m,0} & \cdots & Z_{m,e_m-1} \\ Z_{1,1} & \cdots & Z_{1,e_1} & Z_{2,1} & \cdots & Z_{2,e_2} & \cdots\cdots & Z_{m,1} & \cdots & Z_{m,e_m} \end{bmatrix}.$$

Then the matrix Φ can be considered as the homomorphism of sheaves : $\Phi : F = \oplus^d O_P(-1) \to G = \oplus^2 O_P$. From this homomorphism Φ, we can construct a family of complexes $\mathcal{C}_\bullet^b$ $(b \in \mathbb{Z})$ which resolve the b-th symmetric powers of $Coker\Phi$ by O_P-free modules, and are a kind of generalized Eagon-Northcott complexes, or named as Buchsbaum-Eisenbud complexes associated to Φ (cf. [2]). The complex $\mathcal{C}_\bullet^b = \{\mathcal{C}_i^b, \quad \delta_i : \mathcal{C}_i^b \to \mathcal{C}_{i-1}^b\}$ is :

$$\mathcal{C}_i^b = \begin{cases} \bigwedge^i F \otimes S_{b-i} G & 0 \leq i \leq b \\ \bigwedge^{i+1} F \otimes D_{i-b-1}(G^*) \otimes \bigwedge^2 G^* & i \geq b+1, \end{cases}$$

where $S_j G$ and $D_j(G^*) = (S_j(G))^*$ denote the j-th symmetric power of G and j-th divided power,respectively and $\delta_i = \Phi$ if $i \neq b+1$ and $\delta_{b+1} = \wedge^2 \Phi \in H^0(P, \wedge^2 F^* \otimes \wedge^2 G)$. For integers a and $b \geq -1$, it is known that the complex $\mathcal{C}_\bullet^b(a) = \mathcal{C}_\bullet^b \otimes O_P(a)$ gives a minimal O_P-free resolution of the coherent sheaf $O_W(aH + bR)$.

Now we review classical Brill-Noether theory briefly (cf. [1], [7], [4]). Let C be a non-singular complex non-hyperelliptic projective curve of genus $g \geq 3$. In the Picard variety $Pic^d C$ which parametrizes the line bundles of degree d on C, we consider the Brill-Noether locus :

$$W_d^r := \{\xi \in Pic^d C | deg \xi = d,\ h^0(\xi) \geq r+1\} \subseteq Pic^d C,$$

which is a Zariski closed set. Now we set the Brill-Noether number $\rho := g - (r+1)(g+r-d)$. If $\rho \geq 0$, then $W_d^r \neq \phi$ and each irreducible component of W_d^r has at least the dimension ρ. Moreover, if $2 \leq d \leq g-1$ and $0 < 2r \leq d$, then $dim W_d^r \leq d - 2r - 1$.

If the curve C is generic in its moduli, the curve admits a base point free pencil g_d^1 with $g/2 + 1 \leq d < g/2 + 2$ and no pencil of lower degree.

Remark 3.1 *If $g = 5$, then $\dim W_4^1 = 1$ and $\dim W_3^1 \leq 0$, which implies that any non-hyperelliptic curve C of genus $g = 5$ always has a base point free pencil g_4^1. If g=6, then $0 \leq \dim W_4^1 \leq 1$ and $\dim W_3^1 \leq 0$. Hence, almost every curve C of genus $g = 6$ has a complete base point free pencil g_4^1 except the ones which have only g_3^1's.*

Let us take a canonical curve $X \subset P = \mathbb{P}^{g-1}(\mathbb{C})$ of genus $g \geq 5$ with a base point free complete pencil $g_d^1 = \{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ of degree $d \leq g-1$ on X. By the theorem of Riemann-Roch in the geometric version for an effective divisor D on X :

$$\dim \overline{D} = \deg D - 1 - \dim |D|,$$

where $\overline{D}$ denotes the linear span of D in P, we have $\dim \overline{D}_\lambda = d - 2$. By the theorem of Harris-Bertini (cf. [6], [11]), the set

$$W = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D}_\lambda$$

includes the curve X and is a $(d-1)$-dimensional rational normal scroll of degree $f = g - d + 1$, whose type $S(e_1, \ldots, e_{d-1})$ depends on and determines the dimensions $h^0(X, O_X(iD_\lambda))$ for $i \geq 0$. In particular, we obtain

$$e_1 + \ldots + e_{d-1} = f = g - d + 1, \qquad \frac{2g-2}{d} \geq e_1 \geq \ldots \geq e_{d-1} \geq 0.$$

For $d = 3, 4$, more precise facts are known as in [11].

If $d = 3$, namely the canonical curve curve X is trigonal, then $W = S(e_1, e_2)$ and

$$\frac{2g-2}{3} \geq e_1 \geq e_2 \geq \frac{g-4}{3}.$$

In the resolution of singularities $\varphi : U = P(E) \to W$, the linear equivalence $X \sim 3H - (f-2)R$ holds. For $g \geq 5$, the linear system g_3^1 is unique for the trigonal canonical curve X.

If $d = 4$, then the strict transform of the curve X in the resolution of singularities $\varphi : U = P(E) \to W = S(e_1, e_2, e_3)$ is a complete intersection of two divisors $Y \sim 2H - b_1 R$ and $Z \sim 2H - b_2 R$ with $b_1 + b_2 = f - 2$ and $f - 1 \geq b_1 \geq b_2 \geq -1$.

Further more information in the case $g = 5$ or $g = 6$ can be found in [10]. For example, if we assume that $d = 4$, $g = 6$ and the canonical curve X is generic, then $Y \sim 2H - R$, $Z \sim 2H$ and $W = S(1, 1, 1)$ or $W = S(2, 1, 0)$. This fact will be used in §5.

§4 Proof of the Results on g=5.

Let us give a proof of Theorem 2.1. Take a trigonal canonical curve $X \subset P = \mathbb{P}^4(\mathbb{C})$ of genus $g = 5$ as in the theorem. By Remark 3.1, we have a base point free pencil g^1_4 on X. Then, by the facts in the last part of §3, we see that this pencil g^1_4 induces a singular rational scroll W of dimension 3 : $X \subset W \subset P$, where $W = S(1,1,0)$ or $S(2,0,0)$ and that the curve X is a complete intersection by the 2 effective divisors Y and Z of the 3-fold $U = P(E)$. There are 2 cases : (i) $Y \sim 2H$, $Z \sim 2H$; (ii) $Y \sim 2H - R$, $Z \sim 2H + R$. The first case (i) implies that the curve X is a $(2,2,2)$ complete intersection in P and is not trigonal, which is absurd. Thus we have the case (ii) and $W = S(1,1,0)$. In this case, we see that $\deg Y = Y.H^2 = (2H - R).H^2 = 2 \cdot 2 - 1 = 3$, $\deg Z = (2H + R).H^2 = 2 \cdot 2 + 1 = 5$. It is easy to check that the scheme Y is the Hirzebruch surface $\mathbb{F}_1$ and the curve X has a g^1_3. Now we have a O_U-locally free resolution of O_X and O_Z:

$$\begin{array}{ccccccccc}
0 \longleftarrow & O_X & \longleftarrow & O_U & \longleftarrow & O_U(-2H+R) \oplus O_U(-2H-R) & \longleftarrow & O_U(-4H) & \longleftarrow 0 \\
 & \uparrow & & \| & & \uparrow & & & \\
0 \longleftarrow & O_Z & \longleftarrow & O_U & \longleftarrow & O_U(-2H-R) & \longleftarrow & 0 &
\end{array}$$

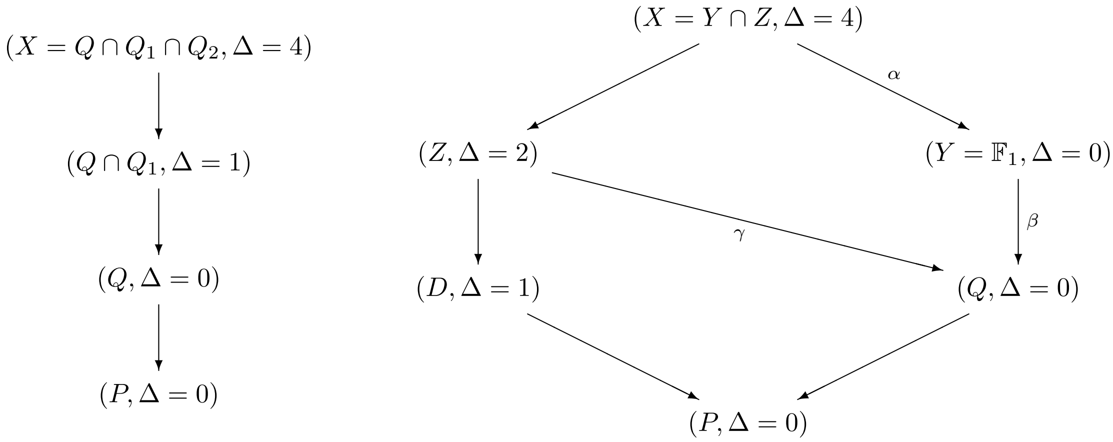
Hence, by taking a multiple mapping cone and a (simple) mapping cone:

$$\begin{array}{c}
\left[\mathcal{C}^0_\bullet \leftarrow \left[\mathcal{C}^1_\bullet(-2) \oplus \mathcal{C}^{-1}_\bullet(-2) \leftarrow \mathcal{C}^0_\bullet(-4)\right]\ \right] \\
\uparrow \\
\left[\mathcal{C}^0_\bullet \leftarrow \mathcal{C}^{-1}_\bullet(-2)\right],
\end{array}$$

we obtain a natural inclusion homomorphism of complexes from a *minimal* O_P-free resolution of O_Z to a *minimal* O_P-free resolution of O_X, which implies that the scheme Z is a homological shell surface of the curve X with degree 5, which means that the scheme Z is irreducible and reduced.

To prove Corollary 2.2, it is enough to show that a non-trigonal canonical curve of $g = 5$ does not have a homological shell surface of degree 5, which is easy and has been already checked in [18].

Let us draw roughly inclusion diagrams for homological shells of the canonical curve X of genus $g = 5$. The diagram in the left hand side is for non-trigonal curves. The one in the right hand side is for the trigonal curves. Arrows in the diagrams denote inclusions. Q's denote quadric hypersurfaces and D denotes a cubic hypersurface. The defect in these diagrams is not to reflect the movements of objects in their families. For example, if we fix the curve X, then the inclusion α or Y is unique. However, the inclusion β is not unique, and the target Q moves in the linear system of dimension 2 whose base locus is Y. On the other hand, when we fix the surface Z, then the inclusion γ or the target Q is unique.



Now it is easy to check the statement of Theorem 2.3 by looking the inclusion diagrams above.

§5 Proof of the Results on g=6.

In this section, we consider a generic canonical curve $X \subset \mathbb{P}^5(\mathbb{C}) = P$ of genus $g = 6$ and prove Theorem 2.4 and Theorem 2.5 simultaneously since each proof does not go separately. Let us start from studying graded Betti numbers of the homological shells of the curve X. As usual, we set $S = \mathbb{C}[Z_0, \cdots, Z_5]$, $S_+ = (Z_0, \cdots, Z_5)S$, and W to be a *good* homological shell of X. By Remark 1.2 (iii), the scheme W is also arithmetically Cohen-Macaulay.

If the curve X is generic, the minimal S-free resolution of the homogeneous coordinate ring R_X is known to be :

$$0 \longleftarrow R_X \longleftarrow S \longleftarrow S(-2)^6 \longleftarrow S(-3)^5 \oplus S(-4)^5 \longleftarrow S(-5)^6$$

$$\longleftarrow S(-7)^1 \longleftarrow 0.$$

From the conditions of homological shells, each term of the minimal S-free resolution of the homogeneous coordinate ring R_W is a direct factor of the corresponding term in the resolution for R_X. Thus we have :

$$0 \longleftarrow R_W \longleftarrow S \longleftarrow S(-2)^{a_1} \longleftarrow S(-3)^{a_2} \oplus S(-4)^{b_2} \longleftarrow S(-5)^{b_3}$$

$$\longleftarrow S(-7)^{b_4} \longleftarrow 0$$

Of course, we have $0 \leq a_1 \leq 6$, $0 \leq a_2 \leq 5$, $0 \leq b_2 \leq 5$, $0 \leq b_3 \leq 6$, and $0 \leq b_4 \leq 1$ for these graded Betti numbers (cf. $a_1 = \beta_{1,2}$, $b_2 = \beta_{2,4}$ etc., where $\beta_{i,j}$ stands for the i-th Betti number in degree j). Now we consider the Hilbert polynomial $A_W(m)$ of the scheme W :

$$A_W(m) = \sum_{k=0}^{N} p_k(W) A_k(m) \qquad A_k(x) := \left(\begin{array}{c} x+k \\ k \end{array} \right).$$

By applying Lemma 2.6 in [14], we can write down the Hilbert polynomial $A_W(m)$ of W which is described by these graded Betti numbers.

$$\begin{cases} p_5 = p_5(W) = 1 - a_1 + a_2 + b_2 - b_3 + b_4 \\ p_4 = p_4(W) = 2a_1 - 3a_2 - 4b_2 + 5b_3 - 7b_4 \\ p_3 = p_3(W) = -a_1 + 3a_2 + 6b_2 - 10b_3 + 21b_4 \\ p_2 = p_2(W) = -a_2 - 4b_2 + 10b_3 - 35b_4 \\ p_1 = p_1(W) = b_2 - 5b_3 + 35b_4 \\ p_0 = p_0(W) = b_3 - 21b_4 \end{cases}$$

For example, if we want to find every homological shell of dimension 1, we have only to solve $p_5 = p_4 = p_3 = p_2 = 0$ and $p_1 > 0$ within the range given above. Then we get $a_1 = 6, \cdots$, which means that the homological shell W is defined by using all the equations of X, namely $W = X$.

Next we consider the case $\dim W = 2$, namely $p_5 = p_4 = p_3 = 0$ and $p_2 > 0$ with the range above. In this case, $d = \deg W = p_2$ and we have two solutions : $(a_1, a_2, b_2, b_3, b_4; d) = (4, 2, 3, 2, 0; 6)$ and $(5, 5, 0, 1, 0; 5)$. In both solutions, we have $b_4 = 0$, which shows $hd_S(R_W) \leq 3$, namely $arith.depth(R_W) \geq 6 - 3 = 3$, and therefore the homogeneous coordinate ring R_W of W is Cohen-Macaulay. In the case $\dim W = 2$, without assuming goodness, the homological shells are automatically good. By the reason that the scheme W includes the curve X which is linearly non-degenerate, we have $h^0(W, O_W(1)) = 6$. Hence, the Δ-genus : $\Delta(W, O_W(1)) = \deg(W) + 2 - h^0(W, O_W(1)) = 2$ or 1.

Now we study the case $\dim W = 3$, or equivalently $p_5 = p_4 = 0$ and $d = \deg W = p_3 > 0$. Solving these equations within the range above, we have 16 solutions including a solution : $(a_1, a_2, b_2, b_3, b_4; d) = (3, 0, 4, 2, 0, 1)$. This solution looks impossible since its degree is too low. However, $b_3 = 2 \neq 0$ means that the scheme W is not arithmetically Cohen-Macaulay and might be non-equidimensional. For example, the scheme W might be the union of a 3-plane and a 2-dimensional scheme including the curve X.

To avoid these complicated situations, here we assume the goodness of the scheme W, namely $b_3 = b_4 = 0$. Then, the scheme W is equidimensional with $\dim W = 3$. We have only two solutions: $(a_1, a_2, b_2, b_3, b_4; d) = (2, 0, 1, 0, 0, 4)$ or $(3, 2, 0, 0, 0, 3)$. Since $a_1 \geq 2$ and the curve X is integral, the scheme W is a subscheme of $(2, 2)$-complete intersection U. The scheme U has $\deg U = 4$. In the case of $\deg W = 4$, $W = U$, where the type of Betti numbers of the scheme W : $(a_1, a_2, b_2, b_3, b_4) = (2, 0, 1, 0, 0)$ coincides with that of $(2, 2)$-complete intersections. We will discuss the integrality of the scheme W later on.

Now we consider the case $(a_1, a_2, b_2, b_3, b_4; d) = (3, 2, 0, 0, 0, 3)$. Take an irreducible component W_0 of W which includes the curve X. Since the curve X is linearly non-degenerate, the scheme $(W_0)_{red}$ with reduced structure has at least degree 3. Therefore $W = (W_0)_{red}$, which means that the scheme W is a variety. Since the variety W is arithmetically Cohen-Macaulay and linearly non-degenerate, we have $h^0(W, O_W(1)) = 6$, which implies that $\Delta(W, O_W(1)) = 3 + 3 - 6 = 0$, namely the variety W is a variety of minimal degree. Hence $W = S(e_1, e_2, e_3)$ with $e_1 + e_2 + e_3 = 3$ and $e_1 \geq e_2 \geq e_3 \geq 0$.

Let us go back to the case $(a_1, a_2, b_2, b_3, b_4) = (2, 0, 1, 0, 0)$. Take again an irreducible component W_0 of W which includes the curve X. Then, it is easy to see that $2 \cdot \deg(W_0)_{red} \geq 6 > \deg W = 4 \geq \deg(W_0)_{red} \geq 3$. Hence we obtain $W_0 = (W_0)_{red}$, namely W_0 is reduced. Assume that $\deg W_0 = 3$. Then the scheme W_0 is a variety of minimal degree and therefore arithmetically Cohen-Macaulay. By using the assumption of a homological shell, from the successive inclusions : $X \subset W_0 \subset W$, we get a natural injective homomorphism : $\mathbb{C}^1 \cong Tor_2^S(R_W, S/S_+)_{(4)} \to Tor_2^S(R_{W_0}, S/S_+)_{(4)} \to Tor_2^S(R_X, S/S_+)_{(4)} \cong \mathbb{C}^5$, where the lower indices "(k)" in right hand side of each term denote taking its degree k-part. By the reason that the variety W_0 is of minimal degree, the homogeneous coordinate ring R_{W_0} has a 2-linear resolution, which shows that $Tor_2^S(R_{W_0}, S/S_+)_{(4)} = 0$, which induces a contradiction. Thus we see that the scheme W is integral.

Now we check the existence of those homological shells. Take a generic canonical curve $X \subset \mathbb{P}^5(\mathbb{C})$ of genus $g = 6$. By Remark 3.1, we see that the curve X has the base point free complete linear system g^1_4 of degree $d = 4$. Recalling the fact in the last part of §3, there is a 3-dimensional rational normal scroll $W = S(1, 1, 1)$ or $W = S(2, 1, 0)$ including the curve X such that in the resolution of singularities $\varphi : U = P(E) \to W$, the strict transform of the curve X is a complete intersection of effective divisors $Y \sim 2H - R$ and $Z \sim 2H$. Then, by using the fact $H^3 = 3$ and the similar calculation as in §4, we have $\deg Y = 5$ and $\deg Z = 6$. Since the variety $W = S(1, 1, 1)$ or $W = S(2, 1, 0)$ is arithmetically Cohen-Macaulay, the divisor Z can be obtain by cutting the variety W with a quadric hypersurface Q', namely $Z = W \cap Q'$. Let us see that both the scheme Y and Z are homological shells of the curve X. By the construction of the curve X, we have a O_U-locally free resolution of O_X and O_Y:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & O_X & \longleftarrow & O_U & \longleftarrow & O_U(-2H+R) \oplus O_U(-2H) & \longleftarrow & O_U(-4H+R) & \longleftarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & & & \\ 0 & \longleftarrow & O_Y & \longleftarrow & O_U & \longleftarrow & O_U(-2H+R) & \longleftarrow & 0 & & \end{array}$$

Hence, by taking a multiple mapping cone and a (simple) mapping cone:

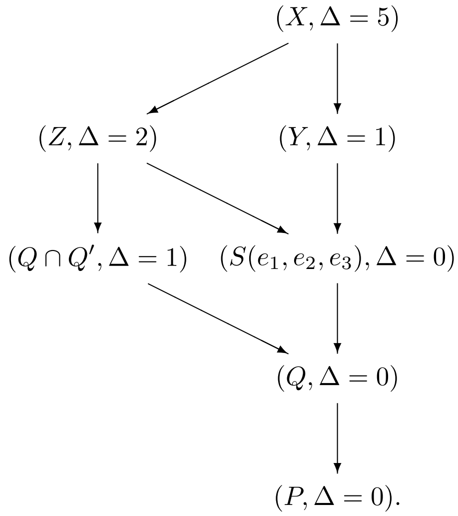
$$\begin{array}{c} \left[\mathcal{C}^0_\bullet \leftarrow \left[\mathcal{C}^1_\bullet(-2) \oplus \mathcal{C}^0_\bullet(-2) \leftarrow \mathcal{C}^1_\bullet(-4)\right]\ \right] \\ \uparrow \\ \left[\mathcal{C}^0_\bullet \leftarrow \mathcal{C}^1_\bullet(-2)\right], \end{array}$$

we obtain a natural inclusion homomorphism of complexes from a *minimal* O_P-free resolution of O_Y to a *minimal* O_P-free resolution of O_X, which implies that the scheme Y is a homological shell of the curve X. We can use the same method to show that the scheme Z is also a homological shell of X. Now we take a quadric hypersurface Q including the variety $W = S(e_1, e_2, e_3)$ where $(e_1, e_2, e_3) = (1, 1, 1)$ or $(e_1, e_2, e_3) = (2, 1, 0)$. Since the variety Q is a homological shell of the variety W, the scheme $Q \cap Q'$ is a homological shell of the scheme $Z = W \cap Q'$. On the other hand, the scheme Z is a homological shell of X. Thus the scheme $Q \cap Q'$ is a homological shell of the curve X.

Let us go back to the general situation of this section. Take two homological shells W and W' of the curve X. Then, their graded Betti numbers $(a_1, a_2, b_2, b_3, b_4)$ and $(a'_1, a'_2, b'_2, b'_3, b'_4)$ satisfy $a_1 \le a'_1$, $a_2 \le a'_2$, $b_2 \le b'_2$, $b_3 \le b'_3$, and $b_4 \le b'_4$ if there is an inclusion $W \supseteq W'$. Hence, by comparing the graded Betti numbers, we can write down the (maximum) inclusion diagram of the homological shells of the curve X. Here we use the word "maximum" to show the set of inclusions is the maximum, namely some of arrows for inclusions might be missing in the real situation. As we saw, this maximum inclusion diagram is realized by the examples in the proof of existence above.

To write down the maximum inclusion diagram, let us denote 2-dimensional homological shell of the curve X with graded Betti numbers and degree : $(a_1, a_2, b_2, b_3, b_4; d) = (5, 5, 0, 1, 0; 5)$ and $(4, 2, 3, 2, 0; 6)$ by Y and Z, respectively.

Then the maximum inclusion diagram is :



Now it is easy to check the statement of Theorem 2.4, namely the monotonously decreasing of the Δ-genera by following arrows from the top X to the bottom P in the maximum inclusion diagram above.

Remark 5.1 *From the view point of classifying* 2*-dimensional homological shells of the curve* X *including trigonal case and plane-quintic case, it looks very important for us to get the claim of Conjecture 1.8. For example, it is not so easy to identify the scheme* $(Z, \Delta = 2)$ *in the inclusion diagram above without this conjecture.*

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(Received October 26, 2012 ; Accepted February 7, 2013)