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Homological shell surfaces with degree 5 of a trigonal canonical curve of genus 5

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Abstract

For a given trigonal canonical curve X of genus 5, we study projective surfaces of degree 5 in $\mathbb{P}^4(\mathbb{C})$ which may have singularities and are "homological shells" (i.e. an alias of "pregeometric shells") of the curve X. In [16], we found the possibilities of their existence and that these surfaces are the homological shells of new type if they exist. Here, in two ways, we construct examples of such surfaces. One way is rather algebraic and to apply the explicit minimal free resolutions exploited in [16], the other one is a more general and geometric way stimulated by an idea of Prof. A. Ohbuchi.
Keywords: surface of degree 5, homological shell, pregeometric shell, canonical curve, trigonal curve, genus 5

§0 Introduction.

In [16], we classified all the homological shells (previously we called them as "pregeometric shells") of a canonical curve X of genus ≤ 5. In the process of this classification, we found interesting phenomena that if the curve X is a non-trigonal canonical curve of genus 5, its homological shells of dimension 2 are always arithmetically Cohen-Macaulay varieties of Δ-genus 1, and that if the curve X is a trigonal canonical curve of genus 5, its homological shells of dimension 2 are always arithmetically Cohen-Macaulay varieties of Δ-genus 0 or Δ-genus 2. In case of Δ-genus 0, it is well known from around a hundred years ago that for a given trigonal canonical curve X of genus 5, such a homological shell surface always exists uniquely and is smooth. On the other hand, if Δ-genus 2, we could not prove their existence, their uniqueness, nor smoothness within our previous work [16].

Frankly speaking, the author expected in the first place that the case of Δ-genus 2 can not occur really. However, Prof. A. Ohbuchi kindly suggested a possibility of this case by constructing a surface of degree 5 containing a trigonal curve of genus 5 as an image of a morphism given by a linear system. But, at the early stage, it was not so easy to analyze explicitly the syzygies (in particular, each differential map of the complex) of the surface. Then, to construct a homological shell surface with Δ-genus 2 of a trigonal canonical curve X of genus 5, the author tried to apply the explicit minimal free resolutions in [16], obtained an example of such a surface by writing down its equations explicitly, and found that the surface is not smooth. Hence we find a fact that contrary to the case of Δ-genus 0, homological shell surfaces with Δ-genus 2 of a trigonal canonical curve X of genus 5 are not smooth in general. After the discovery of this example, we returned to study the surface of degree 5 constructed by Prof. A. Ohbuchi, and found another way of constructing the same surface, which help us to show that the surface is also a homological shell of the curve X (cf. Remark 3.8).

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Thus, in this article, we construct examples of the homological shell surfaces of Δ-genus 2 in two ways described above. Between the two ways, the former one is rather algebraic and the latter one is rather geometric. We expect that each method of construction has an advantage in a different aspect respectively. In these examples, the curve X never be a hypersurface cut of the homological shell surface W because $\deg(X) = 8$, $\deg(W) = 5$. Moreover, the surface W is of Δ-genus 2, which never be a variety of minimal degree. So, these examples are not typical examples of homological shells already known and are a new type of homological shells.

In this article, we use successively the notation and conventions in [16] without mention.

The author would like to express his deep gratitude again to Prof. A. Ohbuchi for his useful suggestion which leads into finding a new type of homological shells.

§1 Preliminaries.

In this section, we make preparations for our argument with summarizing several facts and concepts appeared in our previous papers : [13] $\sim$ [16].

Notation and Conventions 1.1 *For simplicity, an integral projective scheme of dimension* 2 *and an integral projective scheme of dimension* 1 *are called* a surface *and* a curve*, respectively. The both objects may have singularities in general.*

On the other hand, when we say that a non-degenerate closed subscheme X of a projective space is a canonical curve*, it means that there is a non-hyperelliptic smooth projective curve C of genus $g \geq 3$, and the image of the canonical embedding $\Phi_{|K_C|} : C \to \mathbb{P}^{g-1}(\mathbb{C})$ coincides with X. Hence, canonical curves are always smooth.*

To handle Hilbert polynomials efficiently, the Hilbert polynomial $\chi(O_{P^k}(m))$ of $P^k = \mathbb{P}^k(\mathbb{C})$ is written by $A_k(m)$, where

$$A_k(x) := \left(\begin{array}{c} x+k \\ k \end{array} \right) = \frac{(x+k)(x+k-1)\cdots(x+1)}{k!}.$$

For a coherent O_P-module F on $P = \mathbb{P}^N(\mathbb{C})$, its Hilbert polynomial $\chi(F(m)) = A_F(m)$ is expressed in the form:

$$A_F(m) = \sum_{k=0}^{N} p_k(F) A_k(m).$$

For a closed subscheme $V \subseteq P$, we use the symbols: $A_V(m)$ and $p_k(V)$ instead of $A_{O_V}(m)$ and $p_k(O_V)$, respectively.

Remark 1.2 *Since $A_k(m) - A_k(m-1) = A_{k-1}(m)$, we have $A_k(m) = \sum_{j=0}^{k} A_j(m-1)$. For a projective subvariety $V \subset \mathbb{P}^N(\mathbb{C})$ of dimension n, if we write down the Hilbert polynomial of the variety V in the form $A_V(m) = \sum_{k=0}^{n} \chi_k(V) A_k(m-1)$, then we have $\chi_k(V) = \sum_{j=k}^{n} p_j(V)$, in particular, $\chi_{n-1}(V) = p_n(V) + p_{n-1}(V)$. In this case, the sectional genus $g(V, O_V(1)) = 1 - \chi_{n-1}(V) = 1 - (p_n(V) + p_{n-1}(V))$, the Δ-genus $\Delta(V, O_V(1)) = n + \deg V - h^0(V, O_V(1))$, and $\deg V = p_n(V) = \chi_n(V)$. (cf. [5]).*

Remark 1.3 *On the theory of singular curves such as Riemann-Roch theorem and vanishing theorems, we often refer to [7] and exercises in [6]. In particular, for a singular projective curve C, ω_C and $p_a(C)$ denote the dualizing sheaf and the arithmetic genus (or virtual genus) i.e. $h^1(C, O_C)$, respectively.*

Let us recall our key concepts for studying the geometric structures of projective embeddings. The two of them, namely "H-shell" and "G-shell", were introduced first in [13]. We can find many good actual examples of these concepts in a number of classical works in Complex Projective Geometry such as [9], [10], [11], [5] and so on.

Definition 1.4 (shells and cores) *Take a polynomial ring* $S := \mathbb{C}[Z_0, \dots Z_N]$ *of* $(N+1)$*-variables over the complex number field* $\mathbb{C}$ *with the usual grading, and its maximal homogeneous ideal* $S_+ := (Z_0, \dots Z_N)S$*. Let* V *and* W *be closed subschemes of* $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ *which satisfy* $V \subseteq W$ *(namely the inclusion of the defining ideal sheaves:* $I_V \supseteq I_W$ *in the structure sheaf* O_P *of* P*; In this case, the subscheme* W *is called simply an* intermediate ambient scheme *of* V*).*

(1.4.1) *If the natural map:*

$$\mu_{q,x} : Tor_q^{O_{P,x}}(O_{W,x}, O_{P,x}/\mathfrak{m}_{P,x}) \to Tor_q^{O_{P,x}}(O_{V,x}, O_{P,x}/\mathfrak{m}_{P,x})$$

is injective for any integer $q \geq 0$ *and for any point* x *on* V *(abbr.* "local Tor injectivity condition"*), we say that* W *is a* local shell *of* V *and that* V *is a* local core *of* W*, where* $\mathfrak{m}_{P,x}$ *denotes the maximal ideal of the local ring* $O_{P,x}$*.*

(1.4.2) *If the natural map:*

$$\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ *(abbr.* "global Tor injectivity condition"*), we say that* W *is a* homological shell *(abbr.* H-shell*) of* V *and that* V *is a* homological core *(abbr.* H-core*) of* W*, where* $R_W := S/\mathbb{I}_W$ *and* $R_V := S/\mathbb{I}_V$ *denote the homogeneous coordinate rings of* W *and of* V*, respectively, and* $\mathbb{I}_W := \oplus_m H^0(P, I_W(m))$*,* $\mathbb{I}_V := \oplus_m H^0(P, I_V(m))$*.*

(1.4.3) *If the schemes* V *and* W *are closed subvarieties, the variety* W *is a local shell of* V*, and also a homological shell of* V*, then we say that the variety* W *is a* geometric shell *(abbr.* G-shell*) of* V *and the variety* V *is a* geometric core *(abbr.* G-core*) of* W*.*

For the subscheme V*, the total space* P *and* V *itself are called* trivial PG-shells *(or* trivial G-shells *if* V *is a variety).*

Remark 1.5 *Homological shell defined in (1.4.2) above was called as* Pregeometric shell *or* PG-shell *in our previous works after we introduced this concept in [13].*

On geometric shells, we have slightly modified in [15] its definition from the one including the condition: $V \subseteq Reg(W)$ *in [13] to the other one including the condition of local Tor injectivity as in (1.4.3).*

Let us give a review by summarizing our previous results as a theorem. Here we should emphasize again that our "schemes" of course may have a non-equidimensional component or a non-reduced structure.

Theorem 1.6 (cf.[16]) *Let* $X \subseteq P = \mathbb{P}^{g-1}(\mathbb{C})$ *be a canonical curve of genus* $g \leq 5$*. Namely, taking a non hyperelliptic curve* C *of genus* $3 \leq g = g(C) \leq 5$ *and its canonical embedding* $\Phi_{|K_C|} : C \to P = \mathbb{P}^{g-1}(\mathbb{C})$*, we set* $X := \Phi_{|K_C|}(C)$*. Suppose that a closed subscheme* $W \subseteq P$ *is a homological shell of* X*. Then the scheme* W *is always arithmetically Cohen-Macaulay and is a variety, namely irreducible and reduced. Moreover, the inequality :* $\Delta(W, O_W(1)) \leq \Delta(X, O_X(1)) = \Delta(C, K_C) = g - 1$ *holds,* *where the ample line bundles* $O_W(1)$ *and* $O_X(1)$ *are the restrictions of the ample tautological line bundle* $O_P(1) = O_P(H)$.

In case of $g = 5$ *and* $\dim W = 2$, *then* $\Delta(W, O_W(1)) = 1$ *if the curve* X *is non-trigonal,* $\Delta(W, O_W(1)) = 0, 2$ *if the curve* X *is trigonal.*

Moreover, if $\Delta(W, O_W(1)) = 2$ *in the latter case above, then the Hilbert polynomial* $A_W(m)$ *of the surface* W *has the form :* $A_W(m) = 5 \cdot A_2(m) - 6 \cdot A_1(m) + 2$, *which implies that the sectional genus of the surface* W *is* $g(W, O_W(1)) = 2$ *(cf. Remark 1.2). In this case, the homogeneous ideal of the surface* W *is generated minimally by* 1 *quadric equation and* 2 *cubic equations which form a part of minimal generators of the homogeneous ideal of the curve* X, *which is generated minimally by* 3 *quadric equations and* 2 *cubic equations. (cf. [16] (#-3), Table* 1 *)*

From several classical references, let us recall some results which we use later.

Theorem 1.7 (Fujita cf. [5] (3.5)) *Let* (V, L) *be a polarized variety of dimension* n *having a ladder. Assume an inequality on the sectional genus and the* Δ*-genus :* $g(V, L) \geq \Delta(V, L)$. *Then we have*

(1.7.1) *the ladder is regular if* $\deg(V, L) \geq 2\Delta(V, L) - 1$,

(1.7.2) $Bs|L| = \phi$ *if* $\deg(V, L) \geq 2\Delta(V, L)$,

(1.7.3) L *is simply generated,* $g(V, L) = \Delta(V, L)$, *and* $H^q(V, tL) = 0$ *for any* $t, q \in \mathbb{Z}$ *with* $0 < q < n$, *if* $\deg(V, L) \geq 2\Delta(V, L) + 1$. *In this case, the complete linear system* $|L|$ *is very ample, and the embedded variety* $V \subseteq \mathbb{P}^N(\mathbb{C})$ *by the linear system* $|L|$ *is arithmetically Cohen-Macaulay, where* $N = \dim |L|$.

Lemma 1.8 (Castelnuovo bound, cf. e.g. [1]) *Let* $C \subseteq \mathbb{P}^N(\mathbb{C})$ $(N \geq 3)$ *be a non-degenerate smooth projective curve of degree* d *and genus* g. *Take integers* q *and* r *which satisfy* $d - 1 = (N - 1)q + r$ *and* $N - 1 > r \geq 0$. *Then the genus of the curve* C *satisfies the inequality:*

$$g \leq \frac{q(q-1)(N-1)}{2} + qr.$$

Moreover, if the equality above holds, then the curve C *is arithmetically normal.*

The next classical result on an inner projection is well-known roughly in some way, whose rigorous proof can be found in e.g. [4].

Lemma 1.9 *Let* $V \subseteq \mathbb{P}^N(\mathbb{C})$ *be a projective subvariety and* $x_0 \in V$ *a (closed) point. Taking a hyperplane* $H \subset \mathbb{P}^N(\mathbb{C})$ *which does not contain the point* x_0, *we consider a linear projection* $f : \mathbb{P}^N(\mathbb{C}) - \{x_0\} \to H$ *and set the closure of the image* $\overline{f(V - \{x_0\})}$ *to be* W. *Then we have:*

$$\deg f \cdot \deg W = \deg V - e(O_{V,x_0}),$$

where $e(O_{V,x_0})$ *denotes the multiplicity of the local ring* O_{V,x_0}, *which will be denoted simply also by* $e(x_0)$. *In this case, the morphism* $f|_{V^\circ} : V^\circ := V - \{x_0\} \to W$ *is called an* inner projection *of* V *from the center* x_0.

§2 Surfaces of degree 5 and trigonal canonical curves of genus 5

As a preparation for our later argument, without assuming smoothness, let us study a curve of degree 5 in $\mathbb{P}^3(\mathbb{C})$.

Lemma 2.1 *Let $C \subset \mathbb{P}^3(\mathbb{C})$ be a non-degenerate integral closed subscheme of dimension 1 with degree 5. Consider the normalization $\widetilde{C} \to C$ of the curve C. Then the genus $\widetilde{g} = g(\widetilde{C})$ of $\widetilde{C}$ takes the value $\widetilde{g} = 0, 1, 2$. Moreover, even if we assume that the curve C is arithmetically Cohen-Macaulay, the genus $\widetilde{g}$ takes the value in the same range.*

Proof. First we assume that the curve is smooth. Then, $\widetilde{C} = C$, we apply the Castelnuovo bound in Lemma 1.8 and get $\widetilde{g} \leq 2$. Now we suppose that the curve C has a singular point x_0. Take the inner projection $f : C - \{x_0\} \to D \subset H \cong \mathbb{P}^2(\mathbb{C})$ of C from the center x_0. Apply Lemma 1.9, we see that

$$\deg f \cdot \deg D = \deg C - e(x_0) \leq 5 - 2 = 3.$$

Since the curve C is non-degenerate, we have $\deg D \geq 2$ and therefore $\deg f = 1$ and $\deg D = 2, 3$, which means the rational map f is birational. Now the curve D is a divisor of $\mathbb{P}^2(\mathbb{C})$ with degree 2 or 3, which implies that the arithmetic genus takes the value $p_a(D) = 0, 1$. Then the normalization $\widetilde{D} \to D$ of the curve D coincides with $\widetilde{C}$, which shows that $\widetilde{g} = 0, 1$.

Let us construct non-degenerate arithmetically Cohen-Macaulay projective curves of degree 5 with $\widetilde{g} = 0, 1, 2$.

Put $U := \mathbb{P}^1 \times \mathbb{P}^1$ and a morphism $pr_i : U \to \mathbb{P}^1$ $(i = 1, 2)$ to be the projection to the i-th factor. For a point $x \in \mathbb{P}^1$, we set the i-th ruling $h_i := pr_i^{-1}(x)$ on the surface U. If $a, b \in \mathbb{Z}_{\geq 0}$, the linear system $|ah_1 + bh_2|$ has the dimension $ab + a + b$. Now we give an embedding of U into $\mathbb{P}^3(\mathbb{C})$ by using a very ample complete linear system $|h_1 + h_2|$. Then, any effective divisor $D \in |2h_1 + 3h_2|$ has the degree 5 in $\mathbb{P}^3(\mathbb{C})$. Since the canonical divisor K_U of the surface U satisfies $|K_U| = |-2h_1 - 2h_2|$, the adjunction formula shows that the effective divisor D has the arithmetic genus $p_a(D) = 2$ if the divisor D is irreducible and reduced. Now we take two points $q_1, q_2 \in U$ with $pr_i(q_1) \neq pr_i(q_2)$ $(i = 1, 2)$.

It is easy to see that the linear system $\Lambda := |2h_1 + 3h_2 - 2q_1 - 2q_2|$ has no unassigned base point, namely $Bs\Lambda = \{2q_1, 2q_2\}$. In fact, by $\Lambda \supseteq |h_1 + h_2 - q_1 - q_2| + |h_1 + h_2 - q_1 - q_2| + |h_2|$, we have only to show that $Bs|h_1 + h_2 - q_1 - q_2| = \{q_1, q_2\}$. Set divisors $F_1 := pr_1^{-1} \circ pr_1(q_1) + pr_2^{-1} \circ pr_2(q_2)$ and $F_2 := pr_1^{-1} \circ pr_1(q_2) + pr_2^{-1} \circ pr_2(q_1)$. Then the both divisors F_1 and F_2 are the members of the linear system $|h_1 + h_2 - q_1 - q_2|$ and $F_1 \cap F_2 = \{q_1, q_2\}$, which means that $Bs|h_1 + h_2 - q_1 - q_2| = \{q_1, q_2\}$.

We take a blowing up $\mu : \widetilde{U} := B\ell_{q_1,q_2}(U) \to U$ of the surface U at the two points $\{q_1, q_2\}$ and put the exceptional curve e_j to be $\mu^{-1}(q_j)$ $(j = 1, 2)$. Then we see that the linear system $\widetilde{\Lambda} := |2\mu^* h_1 + 3\mu^* h_2 - 2e_1 - 2e_2|$ on the surface $\widetilde{U}$ is base point free. Since $(2\mu^* h_1 + 3\mu^* h_2 - 2e_1 - 2e_2)^2 = 12 - 4 \times 2 = 5 > 0$, Bertini's theorem gives a smooth irreducible member $\widetilde{D_0} \in \widetilde{\Lambda}$. Now we put $D_0 := \mu_*(\widetilde{D_0})$. Then the divisor D_0 is irreducible and reduced and has singularities only at the two points $\{q_1, q_2\}$ as double points. The normalization of the curve D_0 coincides with the smooth curve $\widetilde{D_0}$, whose genus $\widetilde{g} = g(\widetilde{D_0})$ is 0 by the reason that $p_a(D_0) = 2$ and the curve D_0 has only two double points. Since $D_0 \subset U$ and $\deg D_0 = 5 > 2$, the curve D_0 is obviously non-degenerate in $\mathbb{P}^3(\mathbb{C})$.

Now we denote the line bundle on the surface U corresponding to the linear system $|mh_1 + nh_2|$ $(m, n \in \mathbb{Z})$ by $O_U(m, n)$. Then, for a non-negative integer m, we have an exact sequence:

$$0 \to O_U(m-2, m-3) \to O_U(m, m) \to O_{D_0}(m, m) \to 0,$$

which shows the arithmetic Cohen-Macaulayness of the curve D_0 by using the fact $H^1(U, O_U(m-2, m-3)) = 0$ $(m \in \mathbb{Z}_{\geq 0})$ and the arithmetic Cohen-Macaulayness of the surface $U \subset \mathbb{P}^3(\mathbb{C})$.

Similar argument with replacing the linear system $|2h_1 + 3h_2 - 2q_1 - 2q_2|$ by $|2h_1 + 3h_2 - 2q_1|$ and by $|2h_1 + 3h_2|$, respectively, we obtain non-degenerate arithmetically Cohen-Macaulay curves $D_1, D_2 \in |2h_1 + 3h_2|$ of degree 5 such that the curve D_1 has only one double point at q_1 and the curve D_2 has no singularity. Then, their normalizations $\widetilde{D_1}$ and $\widetilde{D_2}$ satisfies $g(\widetilde{D_1}) = 1$ and $g(\widetilde{D_2}) = 2$. ∎

Let us give a birational classification of the non-degenerate projective surface of degree 5 in $\mathbb{P}^4(\mathbb{C})$.

Theorem 2.2 *Let $V \subseteq \mathbb{P}^4(\mathbb{C})$ be a non-degenerate integral closed subscheme of dimension 2 with degree 5. Then the surface V is birational to a scroll over a curve C, or equivalently, to the product $\mathbb{P}^1 \times C$, where the curve C is of genus ≤ 2. Moreover, even if we assume that the surface V is arithmetically Cohen-Macaulay, still we obtain the same results.*

Proof. If the surface V is smooth, then we see that the surface V is birational to the product $\mathbb{P}^1 \times C$ where the curve C is of genus ≤ 1 by the Ionescu's work [8]. More precisely, as in [8], we can also determine the biholomorphic structures in this case. Now we assume that the surface V has a singular point $\{x_0\}$. Then, as in Lemma 1.9, we consider the inner projection $f : V - \{x_0\} \to Y \subset H \cong \mathbb{P}^3(\mathbb{C})$.

If $\dim Y < \dim V$, then, $\dim Y = 1$, $\deg f = 0$, and by Lemma 1.9, we have $e(O_{V,x_0}) = \deg V = 5$ and see that the surface V is a projective cone $Cone_{x_0}(Y)$ over the curve Y with the vertex x_0. Taking a blowing up of the projective cone $V = Cone_{x_0}(Y)$ at the point x_0, we see that the surface V is birational to the product $\mathbb{P}^1 \times Y$. On the other hand, the curve $Y \subset H \cong \mathbb{P}^3(\mathbb{C})$ can be considered as the generic hyperplane section of the surface $V \subset \mathbb{P}^4(\mathbb{C})$, which implies the curve Y is a non-degenerate curve in $H \cong \mathbb{P}^3(\mathbb{C})$ of degree 5. Set the curve C to be a normalization of the curve Y and apply Lemma 2.1, we have an inequality on the genus of the curve C : $g(C) \leq 2$ and a birational map between the surface V and the product $\mathbb{P}^1 \times C$.

Now we assume that $\dim Y = \dim V$. Then, by Lemma 1.9, we have :

$$0 < \deg f \cdot \deg W = \deg V - e(x_0) \leq 5 - 2 = 3.$$

Using non-degeneracy of the surface $V \subseteq \mathbb{P}^4(\mathbb{C})$, we obtain $\deg W \geq 2$, which implies $\deg f = 1$ or equivalently the morphism $f : V - \{x_0\} \to Y$ giving birational equivalence between the surface V and the surface Y. From the inequality above, we have two cases: (i) $e(x_0) = 3$ and $\deg Y = 2$; (ii) $e(x_0) = 2$ and $\deg Y = 3$. In the first case (i), we get the birational equivalence between the surface V and the product $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us consider the second case (ii). If the surface $Y \subseteq H \cong \mathbb{P}^3(\mathbb{C})$ is non-singular, then, by the famous structure theorem on non-singular cubic surfaces (cf. [6] and its reference), the surface Y is a blowing up at generic 6 points of the projective plane $\mathbb{P}^2(\mathbb{C})$, which brings us the birational equivalence between the surface V and the product $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, in the case (ii), we assume that the surface Y has a singular point x_1. we take again an inner projection $h : Y - \{x_1\} \to Z \subset H' \cong \mathbb{P}^2(\mathbb{C})$. By Lemma 1.9 again, we see :

$$\deg h \cdot \deg Z = \deg Y - e(x_1) \leq 3 - 2 = 1.$$

If $\dim Z < \dim Y$, then $\deg h = 0$, $\dim Z = 1$ and $Y = Cone_{x_1}(Z)$. By the same argument above, we get : the curve Z is a cubic curve in $\mathbb{P}^2(\mathbb{C})$; the normalization C of the curve Z has the genus $g(C) = 0, 1$; the surface V is birational to the product $\mathbb{P}^1 \times C$.

The remaining case is $\dim Z = \dim Y = 2$. Then, $Z = H' \cong \mathbb{P}^2(\mathbb{C})$ and $\deg h = 1$, which means that the morphism $h : Y - \{x_1\} \to Z = \mathbb{P}^2(\mathbb{C})$ is birational, and therefore the surface V is birational to the product $\mathbb{P}^1 \times \mathbb{P}^1$.

To construct an arithmetically Cohen-Macaulay surface V of degree 5 which is birational to the product $\mathbb{P}^1 \times C$ with $g(C) = 0, 1, 2$, we have only to take a projective cone of the curve C constructed in the proof of Lemma 2.1. ∎

Remark 2.3 *It is also possible to analyze the biholomorphic structure of the surface V by following the outline of the birational classification given above. This will be done in the forthcoming paper.*

Theorem 2.4 *Let $X \subseteq \mathbb{P}^4(\mathbb{C}) = P = Proj(S)$ be a trigonal canonical curve of genus 5, $V \subsetneq P$ a non-degenerate integral closed subscheme of dimension 2 with degree 5 which contains the curve X. Assume that the surface V has the sectional genus $g(V, O_V(1)) = 2$ (cf. Remark 1.2) and is linearly normal, namely $h^0(V, O_V(1)) = 5$. Then, the surface V is a homological shell of the curve X if and only if the natural product map $\lambda : H^0(V, O_V(1)) \otimes H^0(V, I_{X/V}(2)) \to H^0(V, I_{X/V}(3))$ is surjective, where $I_{X/V}$ denotes the sheaf of defining ideals of X on V.*

Proof. First we assume that the surface V is a homological shell of the curve X. Then, by Theorem 1.6, we see that the surface V is arithmetically Cohen-Macaulay, in particular $H^1(P, I_V(m)) = 0$ for $m \in \mathbb{Z}$, where I_V denotes the sheaf of defining ideals of V in P. From the exact sequence of sheaves of ideals : $0 \to I_V \to I_X \to I_{X/V} \to 0$, we get an exact sequence for any $m \in \mathbb{Z}$:

$$(\#\text{-}1) \qquad 0 \to H^0(P, I_V(m)) \to H^0(P, I_X(m)) \to H^0(V, I_{X/V}(m)) \to 0.$$

We put $\mathbb{I}_V := \oplus_m H^0(P, I_V(m))$, $\mathbb{I}_X := \oplus_m H^0(P, I_X(m))$ and $\mathbb{I}_{X/V} := \oplus_m H^0(P, I_{X/V}(m))$, which can be considered as finite graded S-modules. As we saw in Theorem 1.6, the homogeneous ideal $\mathbb{I}_V$ is generated minimally by 1 quadric equation and 2 cubic equations which form a part of minimal generators of the homogeneous ideal $\mathbb{I}_X$, which is generated minimally by 3 quadric equations and 2 cubic equations. Then the quotient graded S-module $\mathbb{I}_{X/V} = \mathbb{I}_X/\mathbb{I}_V$ obviously generated by the part of degree 2, which means that the natural product map $\lambda : H^0(V, O_V(1)) \otimes H^0(V, I_{X/V}(2)) \to H^0(V, I_{X/V}(3))$ is surjective.

To show the converse direction, let us suppose the surjectivity of the natural product map λ. By the assumption of the degree and the sectional genus of the surface V, the Hilbert polynomial of the surface V has the form $A_V(m) = 5 \cdot A_2(m) - 6 \cdot A_1(m) + c$, where c denotes a certain constant. Since $\deg O_V(1) = 5 \geq 2 \cdot \Delta(V, O_V(1)) + 1 = 2(2+5-5)+1 = 5$, by Theorem 1.7, the surface V is arithmetically Cohen-Macaulay. By Bertini's theorem, we can take a general hyperplane $H \subset \mathbb{P}^4(\mathbb{C}) = P$ such that the curve $Y = V \cap H$ is irreducible and reduced, which is obviously arithmetically Cohen-Macaulay in $H \cong \mathbb{P}^3(\mathbb{C})$ and may have singularities. Since the curve Y is a hyperplane cut of the surface V, the Hilbert polynomial of the curve Y is $A_Y(m) = A_V(m) - A_V(m-1) = 5 \cdot A_1(m) - 6 \cdot A_0(m) = 5m - 1$. In particular, $h^0(O_Y) - h^1(O_Y) = A_Y(0) = -1$ and therefore $p_a(Y) = h^1(O_Y) = 2$. As we said in Remark 1.3, now we can use fully the theory of singular curves in [7].

For a positive integer m, $\deg O_Y(m) = 5m \geq 2p_a(Y) - 1 = 3$, which implies that $H^1(Y, O_Y(m)) = 0$ ($m \in \mathbb{Z}_{\geq 1}$). Hence, $h^0(O_Y(m)) = \chi(O_Y(m)) = 5m - 1$ for $m \in \mathbb{Z}_{\geq 1}$. Let us consider the sheaf of ideals $I_{Y/H}$ defining the curve Y as a space curve in $H \cong \mathbb{P}^3(\mathbb{C}) = Proj(S_H)$, where S_H denotes a polynomial ring of 4 variables over the field of complex numbers. Since the curve Y is arithmetically Cohen-Macaulay, for a positive integer m, $h^0(I_{Y/H}(m)) = h^0(O_H(m)) - h^0(O_Y(m)) = A_3(m) - 5m + 1$. Thus, $h^0(I_{Y/H}(1)) = 0$, $h^0(I_{Y/H}(2)) = 1$, and $h^0(I_{Y/H}(3)) = 6$, which shows $h^0(I_{Y/H}(3)) - 1 \cdot h^0(O_H(1)) = 6 - 4 = 2$. Namely the homogeneous ideal $\mathbb{I}_{Y/H} = \oplus_m H^0(H, I_{Y/H}(m))$ has 1 quadric equation and 2 cubic equations as a part of minimal generators over the polynomial ring S_H. For integers $k = 1, 2, 3$, an exact sequence $0 \to I_{Y/H}(3-k) \to O_H(3-k) \to O_Y(3-k) \to 0$ gives exact sequences: $0 \to H^1(I_{Y/H}(3-1)) \to H^1(O_H(3-1)) = 0$, $0 = H^1(O_Y(3-2)) \to H^2(I_{Y/H}(3-2)) \to H^2(O_H(3-2)) = 0$, and $0 = H^2(O_Y(3-3)) \to H^3(I_{Y/H}(3-3)) \to H^3(O_H(3-3)) = 0$, which implies that the sheaf of ideals $I_{Y/H}$ is 3-regular, namely $H^k(I_{Y/H}(3-k)) = 0$ for $k = 1, 2, 3$. Then the homogeneous ideal $\mathbb{I}_{Y/H}$ has no minimal generators whose degree is greater than 3. Since $\dim S_H = 4$ and the homogeneous coordinate ring R_Y is arithmetically Cohen-Macaulay of dimension 2, we have a minimal graded S_H-free resolution of R_Y with length 2:

$$0 \longleftarrow R_Y \longleftarrow S_H \longleftarrow S_H(-2)^1 \oplus S_H(-3)^2 \longleftarrow S_H(-a) \oplus S_H(-b) \longleftarrow 0,$$

where a and b are integers with $4 \le a \le b$. Here we use $h^0(O_Y(4)) = \chi(O_Y(4)) = 5 \times 4 - 1 = 19$ and get $a = b = 4$. Now we use that the homogeneous coordinate ring R_V of the surface V is arithmetically Cohen-Macaulay, we obtain a minimal graded S-free resolution of R_V :

$$0 \longleftarrow R_V \longleftarrow S \longleftarrow S(-2)^1 \oplus S(-3)^2 \longleftarrow S(-4)^2 \longleftarrow 0.$$

Hence, to show that the surface V is a homological shell of the curve X, we have to check the injectivity of the maps : $\mu_q : Tor_q^S(R_V, S/S_+) \to Tor_q^S(R_X, S/S_+)$ only for the case $q = 1, 2$ (the case $q = 0$ is trivial).

Using that the ring R_V is arithmetically Cohen-Macaulay, we get again $H^1(P, I_V(m)) = 0$ for $m \in \mathbb{Z}$ and the same exact sequence as in (#-1). Now we apply our assumption of the surjectivity of the map λ. Then we see that the quotient graded S-module $\mathbb{I}_{X/V} = \mathbb{I}_X/\mathbb{I}_V$ has no minimal generators of degree 3, which means that all the minimal generators of degree 3 in $\mathbb{I}_V$ form a part of minimal generators of $\mathbb{I}_X$. In other words, we obtain the injectivity of the map :

$$\mu_1 : Tor_1^S(R_V, S/S_+) \to Tor_1^S(R_X, S/S_+),$$

This observation brings an exact commutative diagram:

(#-2)
$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & \mathbb{I}_{X/V} \arrow[l] \arrow[u] & S(-2)^2 \arrow[l, "\gamma_0"'] \arrow[u] & \mathbb{Z}^{(2)}_{X/V} \arrow[l, "\gamma_1"'] \arrow[u] & 0 \arrow[l] \\
0 & \mathbb{I}_X \arrow[l] \arrow[u, "g_0"] & S(-2)^3 \oplus S(-3)^2 \arrow[l, "\beta_0"'] \arrow[u, "g_1"] & \mathbb{Z}^{(2)}_X \arrow[l, "\beta_1"'] \arrow[u, "g_2"'] & 0 \arrow[l] \\
0 & \mathbb{I}_V \arrow[l] \arrow[u, "f_0"] & S(-2)^1 \oplus S(-3)^2 \arrow[l, "\alpha_0"] \arrow[u, "f_1"] & \mathbb{Z}^{(2)}_V = S(-4)^2 \arrow[l, "\alpha_1"] \arrow[u, "f_2"'] & 0 \arrow[l] \\
 & 0 \arrow[u] & 0. \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $\mathbb{Z}^{(2)}_\diamond$ denotes the first syzygy module of the graded S-module $\mathbb{I}_\diamond$, respectively. We should make an easy remark here that by taking the homomorphisms α_0, β_0, and γ_0 suitably, we can suppose that the homomorphism f_1 is a natural (or "trivial") inclusion as a direct factor and the homomorphism g_1 is a natural (or "trivial") projection to a direct factor.

Our remaining problem is to show the injectivity of the map:

$$\mu_2 : Tor_2^S(R_V, S/S_+) \to Tor_2^S(R_X, S/S_+).$$

Let us assume that this map μ_2 is not injective, namely there is a non zero element $\overline{\sigma} \in Ker[\mu_2]$. Now we study the structures of Tor-groups above. $Tor_2^S(R_V, S/S_+) \cong \mathbb{Z}^{(2)}_V \otimes (S/S_+) \cong (S/S_+)^{\oplus 2}_4$, which is a 2-dimensional vector space over the residue field S/S_+ concentrating at its degree 4 part. Thus the element $\overline{\sigma}$ has a non zero homogeneous representative $\sigma \in (\mathbb{Z}^{(2)}_V)_4$ with degree 4, which is a member of the minimal generators of the graded S-module $\mathbb{Z}_V^{(2)}$. To see the structure of $Tor_2^S(R_X, S/S_+)$, let us recall the minimal graded S-free resolution of the homogeneous coordinate ring R_X of the trigonal canonical curve X (cf. e.g.(#-14) and (#-15) in the later section). Then we have $Tor_2^S(R_X, S/S_+) \cong \mathbb{Z}_X^{(2)} \otimes (S/S_+) \cong (S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2}) \otimes (S/S_+) \cong (S/S_+)_4^{\oplus 3} \oplus (S/S_+)_3^{\oplus 2}$ and $0 = \mu_2(\overline{\sigma}) = [f_2(\sigma)] \pmod{(S_+)\mathbb{Z}_X^{(2)}}$. Since $\sigma \in (\mathbb{Z}_V^{(2)})_4$, we get $f_2(\sigma) \in (\mathbb{Z}_X^{(2)})_4 \cap ((S_+)\mathbb{Z}_X^{(2)}) = (S_+)_1 \cdot (\mathbb{Z}_X^{(2)})_3$. Then, there are three elements $\tau_1, \tau_2, \tau_3 \in (\mathbb{Z}_X^{(2)})_3$ which form the minimal generators of the S-module $\mathbb{Z}_X^{(2)}$ in degree 3 and three linear polynomials $a_1, a_2, a_3 \in (S_+)_1$ such that $f_2(\sigma) = a_1 \cdot \tau_1 + a_2 \cdot \tau_2 + a_3 \cdot \tau_3$. By observing (#-14) and (#-15) precisely, the element $f_1 \circ \alpha_1(\sigma) = \beta_1 \circ f_2(\sigma)$ has no component in the part $S(-3)^2$ of the module $S(-2)^3 \oplus S(-3)^2$. By the reason that the homomorphism f_1 is a natural inclusion of a direct factor as we made a remark above, we can conclude that the element $\alpha_1(\sigma)$ has no component in the part $S(-3)^2$ of the module $S(-2)^1 \oplus S(-3)^2$. Since $\alpha_1(\sigma) \neq 0$, there is a non zero homogeneous equation $Q \in S_2$ such that

$$\alpha_1(\sigma) = \begin{bmatrix} Q \\ 0 \\ 0 \end{bmatrix}.$$

Let us describe the homomorphism α_0 by 1×3-matrix $[G, H_1, H_2]$ where $G \in S_2$ and $H_1, H_2 \in S_3$ and $\{G, H_1, H_2\}$ form a minimal generators of $\mathbb{I}_V$. Then, $0 = \alpha_0 \circ \alpha_1(\sigma) = G \cdot Q$, which is a contradiction. ∎

§3 Geometric Construction

In this section, we construct a homological shell surface of degree 5 in $\mathbb{P}^4(\mathbb{C})$ by using geometric approach, which is first suggested by Prof. A. Ohbuchi and improved by us for calculating syzygies (cf. Remark 3.8). We throughout use the notation in the proof of Lemma 2.1 such as $U = \mathbb{P}^1 \times \mathbb{P}^1$, and the ones in [6] such as $|ah_1 + bh_2 - p|$ for a point $p \in U$ which corresponds to the vector space $H^0(U, I_{\{p\}/U} \otimes O_U(a,b)) \subset H^0(U, O_U(a,b))$ given by the sheaf of ideals $I_{\{p\}/U}$ defining the point p on the surface U.

First, based on the fact that $\dim |ah_1 + bh_2| = ab + a + b$ $(a, b \in \mathbb{Z}_{\geq 0})$, we define a concept on the position of points in the surface U as follows, which is a imitation of the similar classical one in Projective Geometry.

Definition 3.1 *Suppose that n points $\{p_1, \dots, p_n\}$ on the surface $U = \mathbb{P}^1 \times \mathbb{P}^1$ are given. Then we say that the n points $\{p_1, \dots p_n\}$ are in* multi-general position*, if they satisfy the two conditions : (1) for any $a, b \in \mathbb{Z}_{\geq 0}$ with $ab + a + b \leq n$ and for $ab + a + b$ points chosen arbitrarily from the n points $\{p_1, \dots p_n\}$, an effective divisor $D \in |ah_1 + bh_2|$ which passes through all these $ab + a + b$ points is unique, ; (2) for any $a, b \in \mathbb{Z}_{\geq 0}$ with $ab + a + b + 1 \leq n$ and for $ab + a + b + 1$ points chosen arbitrarily from the n points $\{p_1, \dots p_n\}$, there is no effective divisor $D \in |ah_1 + bh_2|$ which passes through all these $ab + a + b + 1$ points.*

Remark 3.2 *Take an effective divisor $D \in |ah_1 + bh_2|$ $(a, b \in \mathbb{Z}_{\geq 0})$. Suppose that the divisor D is reducible and decompose into a union of effective divisors $D = D' \cup D''$ or into a sum of effective divisors $D = D' + D''$. Then there are integers $a', a'', b', b'' \in \mathbb{Z}_{\geq 0}$ with $a = a' + a''$ and $b = b' + b''$ such that $D' \in |a'h_1 + b'h_2|$ and $D'' \in |a''h_1 + b''h_2|$. Hence, by taking a suitable number of points in multi-general position, we can control the irreducibility of effective divisors passing through the given points. For example, take 3 points $\{p_1, p_2, p_3\}$ in multi-general position and consider an effective divisor $D_0 \in |h_1 + h_2 - \sum_{i=1}^{3} p_i|$. Then, it is very easy to show that the divisor D_0 is irreducible.*

Lemma 3.3 *For an integer n with $0 \leq n \leq 8$, if we once get n points $\{p_1, \ldots, p_n\}$ in multi-general position on the surface $U = \mathbb{P}^1 \times \mathbb{P}^1$, then, by taking $8-n$ points generically on U, we can make the 8 points $\{p_1, \ldots, p_8\}$ be in multi-general position. In particular, there exist 8 points in multi-general points.*

Proof. Let us translate the condition of multi-general position into a more explicit condition to be handled easily in each case of n. First we solve an equation : $n = ab+a+b$ with the condition: $8 \geq n \geq 0$, $a \geq b \geq 0$, $n, a, b \in \mathbb{Z}$. Then we have solutions: $(n, a, b) = (0,0,0), (1,1,0), (2,2,0), (3,3,0), (3,1,1), (4,4,0), (5,5,0), (5,2,1), (6,6,0), (7,7,0), (7,3,1), (8,8,0), (8,2,2)$. Once we have n points $\{p_1, \ldots, p_n\}$ in multi-general position, the uniqueness of the divisor $D \in |nh_k - \sum_{i=1}^n p_i|$ $(k=1,2)$ as one of the conditions for multi-general position require that $pr_k(p_1), \ldots, pr_k(p_n) \in \mathbb{P}^1$ $(k=1,2)$ are distinct, which will be called "obvious condition" for multi-general position.

If $3 \geq n \geq 0$, then this obvious condition implies the conditions of multi-general position conversely. In fact, if $D \in |h_1 + h_2 - \sum_{i=1}^3 p_i|$, the obvious condition implies the irreducibility of D. Then, this irreducibility and the facts : $(h_1+h_2)^2 = 2$ and $p_1, p_2, p_3 \in D$ show the uniqueness of the divisor in $|h_1 + h_2 - \sum_{i=1}^3 p_i|$.

Hence, for the cases: $8 \geq n \geq 4$, the condition of multi-general position requires an additional condition that the points $\{p_{j(1)}, \ldots, p_{j(n-3)}\}$ are outside of the unique divisor in $|h_1 + h_2 - \sum_{k=1}^3 p_{i(k)}|$ where $\{1, 2, \ldots, n\} = \{i(1), i(2), i(3)\} \cup \{j(1), \ldots, j(n-3)\}$ denotes an arbitrary division.

On the assumptions of obvious condition and of the additional condition for the cases $8 \geq n \geq 4$, if $n = 5$, then the similar argument to the case of $|h_1+h_2|$ shows that the irreducibility and the uniqueness of a divisor in $|2h_1 + h_2 - \sum_{i=1}^5 p_i|$ or in $|h_1 + 2h_2 - \sum_{i=1}^5 p_i|$. Thus for the cases $8 \geq n \geq 6$, the condition of multi-general position requires another additional condition that the points $\{p_{j(1)}, \ldots, p_{j(n-5)}\}$ are outside of the unique divisor in $|2h_1 + h_2 - \sum_{k=1}^5 p_{i(k)}|$ or in $|h_1 + 2h_2 - \sum_{k=1}^5 p_{i(k)}|$ where $\{1, 2, \ldots, n\} = \{i(1), \ldots, i(5)\} \cup \{j(1), \ldots, j(n-5)\}$ denotes an arbitrary division.

On the assumptions of obvious condition and of the two additional conditions for the cases $8 \geq n \geq 4$ and for the cases $8 \geq n \geq 6$, if $n = 7$, then the similar argument to the case of $|h_1 + h_2|$ shows that the irreducibility and the uniqueness of a divisor in $|3h_1 + h_2 - \sum_{i=1}^7 p_i|$ or in $|h_1 + 3h_2 - \sum_{i=1}^7 p_i|$. Hence, if $0 \leq n \leq 7$, the condition of multi-general position is equivalent to the assembly of the obvious conditions and two additional conditions.

Thus we can prove step by step our claim in Lemma 3.3 except the case $n = 8$. The first point p_1 can be chosen arbitrarily. Once we obtain the first point p_1, it is enough to choose the second point p_2 outside the closed set $Y_1 := pr_1^{-1} \circ pr_1(p_1) \cup pr_2^{-1} \circ pr_2(p_1)$. When we have two points $\{p_1, p_2\}$ in multi-general position, we have only to choose the third point p_3 outside the closed set $Y_2 := \cup_{i,j} pr_i^{-1} \circ pr_i(p_j)$ $(i = 1, 2$ and $j = 1, 2)$. Suppose that three points $\{p_1, p_2, p_3\}$ in multi-general position are given. Then, we have only to choose the fourth poit p_4 outside the closed set $D_1 \cup Y_3$ where $Y_3 := \cup_{i,j} pr_i^{-1} \circ pr_i(p_j)$ $(i = 1, 2$ and $j = 1, 2, 3)$ and D_1 denotes the unique divisor in $|h_1 + h_2 - \sum_{i=1}^3 p_i|$. Assume that four points $\{p_1, \ldots, p_4\}$ in multi-general position are given. Then, the fifth point p_5 has to be chosen outside the closed set $D_2 \cup Y_4$ where $Y_4 := \cup_{i,j} pr_i^{-1} \circ pr_i(p_j)$ $(i = 1, 2$ and $j = 1, \ldots, 4)$ and D_2 denotes the union $\cup D_2(i(1), i(2), i(3))$ $(\{1, 2, 3, 4\} = \{i(1), i(2), i(3)\} \cup \{j(1)\}$ arbitrary division of the set $)$ of the unique divisor $D_2(i(1), i(2), i(3)) \in |h_1 + h_2 - \sum_{k=1}^3 p_{i(k)}|$. Continuing the similar argument, we obtain our claim except for the case $n = 8$. Before we proceed to the case $n = 8$, we have to show the next Lemma 3.4 for the case $a = 1, 2$. Then, based on Lemma 3.4 for the case $a = 1, 2$, we can prove Lemma 3.5.

Let us consider the case $n = 8$. The condition of multi-general position requires at least one more condition that the point $\{p_{j(1)}\}$ is outside of the unique divisor in $|3h_1 + h_2 - \sum_{k=1}^7 p_{i(k)}|$ or in $|h_1 + 3h_2 - \sum_{k=1}^7 p_{i(k)}|$ where $\{1, 2, \ldots, 8\} = \{i(1), \ldots, i(7)\} \cup \{j(1)\}$ denotes an arbitrary division. However, this is not enough to get conversely the condition of multi-general position. Now we assume that 7 points $\{p_1, \ldots, p_7\}$ in multi-general position are given. Then, by Lemma 3.5, we see that $\dim |2h_1 + 2h_2 - \sum_{i=1}^7 p_i| = 1$ and this linear system has one unassigned base point q, namely $Bs(|2h_1 + 2h_2 - \sum_{i=1}^7 p_i|) = \{p_1, \ldots, p_7, q\}$. Thus, to choose the 8-th point p_8, we have to take it not only outside of the divisors coming from the obvious condition and three additional conditions but also avoiding the unassigned base point q. Finally we make a remark that based on these fact we get Lemma 3.4 also for the case $a = 3$. ∎

Lemma 3.4 *For an integer with $a = 1, 2, 3$, suppose that $2a$ points $\{p_1, \dots, p_{2a}\}$ on the surface $U = \mathbb{P}^1 \times \mathbb{P}^1$ in multi-general position are given. Then the linear system $|ah_1 + h_2 - \sum_{i=1}^{2a} p_i|$ has no unassigned base point, namely the base locus of the linear system is exactly $Bs|ah_1 + h_2 - \sum_{i=1}^{2a} p_i| = \{p_1, \dots, p_{2a}\}$. The similar result also holds for the linear system $|h_1 + ah_2 - \sum_{i=1}^{2a} p_i|$ or for the case that the point p_2 is an infinitely near point of the point p_1.*

Proof. The point to be considered is that a member of $|ah_1 + h_2 - \sum_{i=1}^{2a} p_i|$ is not always irreducible. By Lemma 3.3, let us take two points p_{2a+1} and p_{2a+2} of the surface U such that the $2a + 2$ points $\{p_1, \dots, p_{2a+2}\}$ are in multi-general position. Then, by the definition of multi-general position, for $2a+1$ points $\{p_{i(1)}, \dots, p_{i(2a+1)}\}$ arbitrarily chosen from $\{p_1, \dots, p_{2a+2}\}$, $\dim |ah_1 + h_2 - \sum_{j=1}^{2a+1} p_{i(j)}| = 0$ and the unique effective divisor $D \in |ah_1 + h_2 - \sum_{j=1}^{2a+1} p_{i(j)}|$ is obviously irreducible. Hence $\dim |ah_1 + h_2 - \sum_{i=1}^{2a} p_i| = 1$. Let us take two members $D_1 \in |ah_1 + h_2 - p_{2a+1} - \sum_{i=1}^{2a} p_i|$ and $D_2 \in |ah_1 + h_2 - p_{2a+2} - \sum_{i=1}^{2a} p_i|$. Then obviously $D_1 \neq D_2$ and they form a basis of the linear system $|ah_1 + h_2 - \sum_{i=1}^{2a} p_i|$. The divisors D_1 and D_2 are irreducible, $\{p_1, \dots, p_{2a}\} \subseteq D_1 \cap D_2$, and $D_1.D_2 = (ah_1 + h_2)^2 = 2a$. Thus $Bs|ah_1 + h_2 - \sum_{i=1}^{2a} p_i| = D_1 \cap D_2 = \{p_1, \dots, p_{2a}\}$. ∎

Lemma 3.5 *Assume that on the surface $U = \mathbb{P}^1 \times \mathbb{P}^1$, given 6 points $\{p_1, \dots, p_6\}$ are in multi-general position. Then the linear system $|2h_1 + 2h_2 - \sum_{i=1}^{6} p_i|$ has no unassigned base point. In particular,* $\dim |2h_1 + 2h_2 - \sum_{i=1}^{6} p_i| = 2$.

Proof. By the definition of multi-general position, for 3 points $\{p_{i(1)}, p_{i(2)}, p_{i(3)}\}$ arbitrarily chosen from $\{p_1, \dots, p_6\}$, $\dim |h_1 + h_2 - p_{i(1)} - p_{i(2)} - p_{i(3)}| = 0$ and the unique effective divisor $C \in |h_1 + h_2 - p_{i(1)} - p_{i(2)} - p_{i(3)}|$ is irreducible. Let us denote this divisor C as $C(i(1), i(2), i(3))$. Now we take 4 divisors in $|2h_1 + 2h_2 - \sum_{i=1}^{6} p_i|$ as follows.

$$\begin{array}{lll} D_1 := C_1' + C_1'' \,; & C_1' := C(1,2,3) & C_1'' := C(4,5,6) \\ D_2 := C_2' + C_2'' \,; & C_2' := C(1,2,4) & C_2'' := C(3,5,6) \\ D_3 := C_3' + C_3'' \,; & C_3' := C(1,2,5) & C_3'' := C(3,4,6) \\ D_4 := C_4' + C_4'' \,; & C_4' := C(1,2,6) & C_4'' := C(3,4,5) \end{array}$$

Since $\{p_1, \dots, p_6\} \subseteq Bs|2h_1 + 2h_2 - \sum_{i=1}^{6} p_i| \subseteq D_1 \cap D_2 \cap D_3 \cap D_4$, it is enough to show that $D_1 \cap D_2 \cap D_3 \cap D_4 = \{p_1, \dots, p_6\}$.

Let us assume that $D_1 \cap D_2 \cap D_3 \cap D_4 \supseteq \{p_1, \dots, p_6, q\}$. It is easy to see that $\{C_i', C_i'' | i = 1,2,3,4\}$ are distinct 8 curves and $\#(C_i' \cap C_j') = \#(C_i'' \cap C_j'') = \#(C_i' \cap C_j'') = 2$ for $i \neq j$ by the reason that $(h_1 + h_2)^2 = 2$. Then, for $i \neq j$, obviously $C_i' \cap C_j' = \{p_1, p_2\}$, and $C_i'' \cap C_j'' \subset \{p_3, \dots, p_6\}$. Since for $i \neq j$, $D_i \cap D_j \supseteq \{p_1, \dots, p_6, q\}$ and $D_i \cap D_j = (C_i' \cap C_j') \cup (C_i'' \cap C_j'') \cup (C_i' \cap C_j'') \cup (C_j' \cap C_i'')$, the set $(C_i' \cap C_j'')$ or the set $(C_j' \cap C_i'')$ includes the point q. Let us list up all the cases in the following table.

Cases	$C'_i \cap C''_j$	$C'_j \cap C''_i$
(i,j)=(1,2)	$p_3,\ s_1$	$p_4,\ t_1$
(1,3)	$p_3,\ s_2$	$p_5,\ t_2$
(1,4)	$p_3,\ s_3$	$p_6,\ t_3$
(2,3)	$p_4,\ s_4$	$p_5,\ t_4$
(2,4)	$p_4,\ s_5$	$p_6,\ t_5$
(3,4)	$p_5,\ s_6$	$p_6,\ t_6$

Table 1: Intersection Points (I)

Then for each $k = 1, \ldots 6$, s_k or t_k has to be the point q. On the other hand, the case p_ℓ, q does not appear in Table 1 more than once. In fact, if $(i_1, j_1) \neq (i_2, j_2)$ and $C'_{i_1} \cap C''_{j_1} = C'_{i_2} \cap C''_{j_2} = \{p_\ell, q\}$, then $q \in C'_{i_1} \cap C'_{i_2}$ and $q \in C''_{j_1} \cap C''_{j_2}$. Since $(i_1, j_1) \neq (i_2, j_2)$, we have $i_1 \neq i_2$ or $j_1 \neq j_2$. Then we see that $\#C'_i \cap C'_j \geq 3$ or $\#C''_i \cap C''_j \geq 3$ for $i \neq j$, which is a contradiction.

Let us start form the case $s_1 = q$. Then $t_2 = t_3 = s_4 = q$ and therefore $s_5 \neq q$ and $t_5 \neq q$. This is a contradiction. The remaining case is $t_1 = q$. Then $t_4 = t_5 = s_2 = q$ and therefore $s_3 \neq q$ and $t_3 \neq q$. This is also a contradiction. ∎

Lemma 3.6 *Take* 7 *points* $Z := \{p_1, \ldots, p_7\}$ *in multi-general position on the surface* $U = \mathbb{P}^1 \times \mathbb{P}^1$*. Set* $\rho : V := B\ell_Z(U) \to U$ *to be a blowing up of the surface* U *at the center* Z *and the exceptional curve* e_i *to be* $\rho^{-1}(p_i)$ $(i = 1, 2, \ldots, 7)$*. Then, the linear system* $|2\rho^*h_1 + 3\rho^*h_2 - \sum_{i=1}^7 e_i|$ *on the surface* V *is very ample, which embeds* V *as the surface of degree* 5*.*

Proof. For integers : $a, b \in \mathbb{Z}_{\geq 0}$, $1 \leq k(1) < k(2) < \cdots < k(t) \leq 7$ and a point p on the surface $U = \mathbb{P}^1 \times \mathbb{P}^1$ with including a infinitely near point, we put the linear systems:

$$\begin{aligned} \Lambda((a,b); k(1), k(2), \cdots, k(t)) &:= |ah_1 + bh_2 - \textstyle\sum_{j=1}^t p_{k(j)}| \\ \Lambda((a,b); k(1), k(2), \cdots, k(t); p) &:= |ah_1 + bh_2 - p - \textstyle\sum_{j=1}^t p_{k(j)}|. \end{aligned}$$

By using the similar argument to the one for the cubic surfaces in [6], not on the surface V but on the surface U, set $\Lambda := \Lambda((2,3); 1, \cdots, 7)$, we have only to show that, on the set of base points of linear systems, $Bs\Lambda = Z$, and for any point $p \in U$ with including a infinitely near point, $Bs(\Lambda - p) = Z \cup \{p\}$, where $\Lambda - p$ denotes simply the linear system $\Lambda((2,3); 1, \cdots, 7; p)$. Since obviously $Bs\Lambda \supseteq Z$ and $Bs(\Lambda - p) \supseteq Z \cup \{p\}$, it is enough to show the inclusions in the converse direction.

(i) Let us show that $Bs\Lambda \subseteq Z$. For $j = 1, \cdots, 7$, take an effective divisor $L(j) \in |h_2 - p_j|$. Then we get $L(j) + \Lambda((2,2); 1, \cdots \overset{j}{\smile} \cdots, 7) \subseteq \Lambda$. By Lemma 3.5, $Bs(\Lambda((2,2); 1, \cdots \overset{j}{\smile} \cdots, 7)) = \{p_1, \cdots \overset{j}{\smile} \cdots, p_7\}$, we have $L(j) \cup \{p_1, \cdots \overset{j}{\smile} \cdots, p_7\} = Bs(L(j) + \Lambda((2,2); 1, \cdots \overset{j}{\smile} \cdots, 7) \supseteq Bs(\Lambda)$. Hence we obtain

$$Z = \{p_1, \cdots, p_7\} = \bigcap_{j=1}^{7} (L(j) \cup \{p_1, \cdots \overset{j}{\smile} \cdots, p_7\}) \supseteq Bs(\Lambda)$$

(ii) Now we show that $Bs(\Lambda - p) \subseteq Z \cup \{p\}$ for any point $p \in U$ with including a infinitely near point. By the fact $\dim |2h_1 + 2h_2| = 8$, we can take an effective divisor $C_0 \in \Lambda((2,2); 1, \cdots, 7; p)$. By the reason that the points $\{p_1, \ldots, p_7\}$ are in multi-general position, and $p_1, \ldots, p_7 \in C_0$, the divisor C_0 is irreducible. Using $Bs|h_2| = \phi$ and $C_0 + |h_2| \subseteq \Lambda - p$, we see that $Bs(\Lambda - p) \subseteq Bs(C_0 + |h_2|) = C_0$. Let us divide this case into the following three cases.

(ii-a) Suppose that $\Lambda((1,0);1;p) \neq \phi$. Take an effective divisor $L_1 \in \Lambda((1,0);1;p)$. Then $L_1 + \Lambda((1,3);2,\cdots,7) \subseteq \Lambda - p$, which implies that $L_1 \cup \{p_2, \cdots p_7\} = Bs(L_1 + \Lambda((1,3);2,\cdots,7)) \supseteq Bs(\Lambda - p)$ by applying Lemma 3.4. Then we see that $Z \cup \{p\} = C_0 \cap (L_1 \cup \{p_2, \cdots p_7\}) \supseteq Bs(\Lambda - p)$.

(ii-b) Assume that $\Lambda((0,1);1;p) \neq \phi$. Pick an effective divisor $L_2 \in \Lambda((0,1);1;p)$. Then $L_2 + \Lambda((2,2);2,\cdots,7) \subseteq \Lambda - p$, which implies that $L_2 \cup \{p_2, \cdots p_7\} = Bs(L_2 + \Lambda((2,2);2,\cdots,7)) \supseteq Bs(\Lambda - p)$ by applying Lemma 3.5. Hence we have $Z \cup \{p\} = C_0 \cap (L_2 \cup \{p_2, \cdots p_7\}) \supseteq Bs(\Lambda - p)$.

(ii-c) In this remaining case, we can assume that $\Lambda((1,0);1;p) = \phi$.and $\Lambda((0,1);1;p) = \phi$. Hence we see that any effective divisor in $\Lambda((1,1);1;p)$ is irreducible. To show $Z \cup \{p\} \supseteq Bs(\Lambda - p)$, we assume that there is a point x such that $Z \cup \{p, x\} \subseteq Bs(\Lambda - p)$ and deduce the contradiction from this assumption.

Let us divide the set $\{2, \ldots, 7\}$ into two disjoint sets $\{i(1), i(2), \ldots i(5)\}$ and $\{k\}$, and take effective divisors $C_1(i(1), i(2), \ldots i(5)) \in \Lambda((1,2) : i(1), i(2), \ldots i(5))$ and $C_1'(k) \in \Lambda((1,1);1,k;p)$. Since the points $\{p_{k(1)}, \ldots, p_{k(5)}\}$ are in multi-general position, the divisor $C_1(i(1), i(2), \ldots i(5))$ is also irreducible. Moreover, $C_1(i(1), i(2), \ldots i(5)) + C_1'(k) \in \Lambda - p$. For integers $j = 2, \cdots, 7$, we take 6 effective divisors $D_j = F_j + F_j' \in \Lambda - p$ as follows.

$$
\begin{aligned}
F_2 &:= C_1(3,4,5,6,7), & F_2' &:= C_1'(2)\\
F_3 &:= C_1(2,4,5,6,7), & F_3' &:= C_1'(3)\\
F_4 &:= C_1(2,3,5,6,7), & F_4' &:= C_1'(4)\\
F_5 &:= C_1(2,3,4,6,7), & F_5' &:= C_1'(5)\\
F_6 &:= C_1(2,3,4,5,7), & F_6' &:= C_1'(6)\\
F_7 &:= C_1(2,3,4,5,6), & F_7' &:= C_1'(7)
\end{aligned}
$$

By using the irreducibility of the divisors C_0, F_j and F_j' and the intersection numbers $C_0 \cdot F_j = 6$ and $C_0 \cdot F_j' = 4$, we obtain the following table of intersection points.

Cases	$C_0 \cap F_j$	$C_0 \cap F_j'$
D_2	$p_3, p_4, p_5, p_6, p_7, u_2$	p_1, p, p_2, v_2
D_3	$p_2, p_4, p_5, p_6, p_7, u_3$	p_1, p, p_3, v_3
D_4	$p_2, p_3, p_5, p_6, p_7, u_4$	p_1, p, p_4, v_4
D_5	$p_2, p_3, p_4, p_6, p_7, u_5$	p_1, p, p_5, v_5
D_6	$p_2, p_3, p_4, p_5, p_7, u_6$	p_1, p, p_6, v_6
D_7	$p_2, p_3, p_4, p_5, p_6, u_7$	p_1, p, p_7, v_7

Table 2: Intersection Points (II)

By the fact $D_j \cap C_0 \supseteq Bs(\Lambda - p) \supseteq Z \cup \{p, x\}$, we see that $u_j = x$ or $v_j = x$ for any $j = 2, \cdots, 7$. On the other hand, for $2 \leq i < j \leq 7$, if we have $u_i = u_j = x$, then $F_i = F_j$ since the divisors F_i and F_j are irreducible, $F_i \cdot F_j = 4$ and $F_i \cap F_j \supseteq \{p_2 \cdots \overset{i}{\smile} \cdots \overset{j}{\smile} \cdots p_7, x\}$, which implies that $F_i = F_j \supseteq \{p_2, \cdots p_7\}$. This is a contradiction because 6 points in multi-general position are on the effective divisor $F_i = F_j \in |h_1 + 2h_2|$. Thus we see that at least 5 points among the 6 points $\{v_2, \cdots, v_7\}$ coincide with the point x. By change the numbering of the points $\{p_2, \cdots, p_7\}$, we may assume that $v_3 = v_4 = \cdots = v_7 = x$. Then, for $3 \leq i < j \leq 7$, $F_i' = F_j'$ since the divisors F_i' and F_j' are irreducible, $F_i' \cdot F_j' = 2$ and $F_i' \cap F_j' \supseteq \{p_1, p, x\}$. Hence we get $F_3' = \cdots = F_7' \supseteq \{p_3, \cdots p_7\}$. This is also a contradiction because 5 points in multi-general position are on an effective divisor in $|h_1 + h_2|$. ∎

Based on these preparation above, let us construct a pair of a trigonal canonical curve X and a homological shell surface V with degree 5 of the curve X in $\mathbb{P}^4(\mathbb{C})$.

Theorem 3.7 *Suppose the same situation as in Lemma 3.6 above. Take a non-singular curve* $C \in |4h_1 + 5h_2 - \sum_{i=1}^{7} 2\cdot e_i|$ *suitably on the surface* V*. Then we have the following two facts.*

(3.7.1) *The curve C is trigonal and of genus 5. The trace $\Lambda|_C$ of the linear system Λ on the curve C is complete and coincides with the canonical linear system $|K_C|$.*

(3.7.2) *Embed the surface V and the curve C by the linear system Λ, then the curve C is a trigonal canonical curve X and the embedded surface V is a non-singular homological shell of the curve X and of degree 5.*

Proof. First we consider the linear system $|2h_1 + 2h_2 - \sum_{i=1}^{7} p_i|$ on the surface U. Then this linear system $|2h_1 + 2h_2 - \sum_{i=1}^{7} p_i|$ forms a pencil by Lemma 3.5, which has only one unassigned base point p_0, which might be an infinitely near point to one of the 7 points $Z = \{p_1, \ldots, p_7\}$. In any case, we can consider the point p_0 is on the surface V.

Now we take a blowing up $\sigma : \widetilde{V} := B\ell_{p_0}(V) \to V$ at the center p_0 and set $e_0 := \sigma^{-1}(p_0)$. Then, obviously the linear system $|2h_1 + 2h_2 - e_0 - \sum_{i=1}^{7} e_i|$ on the surface $\widetilde{V}$ is a pencil without base point. By Lemma 3.6, the linear system $|2h_1 + 3h_2 - \sum_{i=1}^{7} e_i|$ is very ample on the surface V, and therefore its pullback to the surface $\widetilde{V}$ is base point free. Thus we have a base point free linear system $|4h_1+5h_2-e_0-\sum_{i=1}^{7} 2\cdot e_i|$ on the surface $\widetilde{V}$, whose self intersection is $(4h_1+5h_2-e_0-\sum_{i=1}^{7} 2\cdot e_i)^2 = 40-1-2\times 7 = 25 > 0$. By Bertini's theorem, we obtain a non-singular irreducible curve $\widetilde{C} \in |4h_1 + 5h_2 - e_0 - \sum_{i=1}^{7} 2\cdot e_i|$ on the surface $\widetilde{V}$.

Since the intersection number $\widetilde{C}\cdot e_0 = 1$, the curve $C := \sigma(\widetilde{C})$ is also a non-singular irreducible curve passing simply the point p_0 on the surface V and is a member of the linear system $|4h_1 + 5h_2 - (\sum_{i=1}^{7} 2\cdot e_i) - p_0|$. Obviously the curve $\widetilde{C}$ is the strict transform of the curve C and is isomorphic to the curve C via the morphism σ.

Now let us consider the trace of the base point free pencil $|2h_1 + 2h_2 - e_0 - \sum_{i=1}^{7} e_i|$ on the surface $\widetilde{V}$ to the curve $\widetilde{C}$. Then its degree is $(2h_1 + 2h_2 - e_0 - \sum_{i=1}^{7} e_i).\widetilde{C} = (2h_1 + 2h_2 - e_0 - \sum_{i=1}^{7} e_i).(4h_1 + 5h_2 - e_0 - \sum_{i=1}^{7} 2\cdot e_i) = 18 - 1 - 2\times 7 = 3$, which shows that the curve C is trigonal.

By using adjunction formula, we have $2g(\widetilde{C}) - 2 = (K_{\widetilde{V}} + \widetilde{C}).\widetilde{C} = ((-2h_1 - 2h_2 + e_0 + \sum_{i=1}^{7} e_i) + (4h_1 + 5h_2 - e_0 - \sum_{i=1}^{7} 2\cdot e_i)).(4h_1 + 5h_2 - e_0 - \sum_{i=1}^{7} 2\cdot e_i) = -3 + 40 - 1 - 4\times 7 = 8$, and therefore $g(C) = g(\widetilde{C}) = 5$.

Now we consider the dimension of the linear system $\Lambda = |2h_1 + 3h_2 - \sum_{i=1}^{7} e_i|$. It is easy to see that $\dim\Lambda \geq 11 - 7 = 4$. Take an effective divisor $H \in \Lambda$ and consider an exact sequence : $0 \to O_V(H - C) \to O_V(H) \to O_C(H|_C) \to 0$. Then we have a linear equivalence of divisors on the surface V : $K_V + C \sim (-2h_1 - 2h_2 + \sum_{i=1}^{7} e_i) + (4h_1 + 5h_2 - \sum_{i=1}^{7} 2\cdot e_i)) \sim H$, which brings an exact sequence: $0 \to O_V(K_V) \to O_V(H) \to O_C(K_C) \to 0$ and its induced exact sequence : $0 \to H^0(V, O_V(K_V)) \to H^0(V, O_V(H)) \to H^0(C, O_C(K_C))$. By birational invariance of the geometric genus, $h^0(V, O_V(K_V)) = p_g(V) = p_g(U) = 0$, which implies that $h^0(V, O_V(H)) \leq h^0(C, O_C(K_C)) = 5$ and therefore $4 \leq \dim\Lambda = h^0(V, O_V(H)) - 1 \leq h^0(C, O_C(K_C)) - 1 = 5 - 1$, namely $\dim\Lambda = 4$ and $\Lambda|_C = |K_C|$. Hence we see the claim (3.7.1) holds and that the very ample linear system Λ gives an embedding of the surface V into $\mathbb{P}^4(\mathbb{C})$ as a surface of degree 5, whose restriction to the curve C coincides with its canonical embedding.

To get the claim (3.7.2), we have only to show that the embedded surface V is a homological shell of the canonical curve $C \cong X \subset \mathbb{P}^4(\mathbb{C})$.

Let us recall Theorem 2.4 and check all the conditions for being homological shell. Take a smooth irreducible member $H \in \Lambda$. Then, the sectional genus $g(V, O_V(H)) = g(H)$. By adjunction formula,

$2g(H)-2=(K_V+H).H=((-2h_1-2h_2+\sum_{i=1}^7 e_i)+(2h_1+3h_2-\sum_{i=1}^7 e_i)).(2h_1+3h_2-\sum_{i=1}^7 e_i)=h_2.(2h_1+3h_2-\sum_{i=1}^7 e_i)=2$, we get $g(V,O_V(H))=g(H)=2$.

The remaining condition to be confirmed is that the map $\lambda : H^0(V,O_V(1))\otimes H^0(V,I_{X/V}(2))\to H^0(V,I_{X/V}(3))$ is surjective. Since $I_{X/V}=O_V(-C)$, we have an isomorphism of sheaves: $I_{X/V}(2)\cong O_V(2H-C)\cong O_V(h_2)$. Now we consider the composition of the morphisms $\tau := pr_2\circ\rho : V\to U\to\mathbb{P}^1$. Then, all the global sections in $H^0(V,I_{X/V}(2))=H^0(V,O_V(h_2))$ come from $H^0(\mathbb{P}^1,O_{\mathbb{P}^1}(1))$ on the second factor of the product space U. Pulling back the Euler sequence on the second factor $\mathbb{P}^1$ by the morphism τ and by the morphism $pr_2 : U\to\mathbb{P}^1$, we have two short exact sequences as we will see precisely in the sequel.

The first short exact sequence obtained by τ is $0\to O_V(-h_2)\to H^0(V,O_V(h_2))\otimes O_V\to O_V(h_2)\to 0$. Let us tensor the line bundle $O_V(H)$ to this sequence and get

(#-3)
$$0\longrightarrow O_V(H-h_2)\longrightarrow H^0(V,I_{X/V}(2))\otimes O_V(H)\longrightarrow O_V(H+h_2)\longrightarrow 0.$$

Take its cohomology long exact sequence, we obtain the desired map λ.

(#-4)
$$H^0(V,I_{X/V}(2))\otimes H^0(V,O_V(1))\xrightarrow{\lambda} H^0(V,I_{X/V}(3))\longrightarrow H^1(V,O_V(H-h_2)).$$

Thus, it is enough to show the exactness of the H^0 part of the short sequence (#-3).

Let us consider the second short exact sequence obtained by pr_2. Tensor the sheaf of ideals $I_{Z/U}(2h_1+3h_2)$ defining the 7 points $Z=\{p_1,\ldots,p_7\}$ to this short exact sequence of vector bundles. Then, we obtain the sequence :

(#-5)
$$0\longrightarrow I_Z(2h_1+2h_2)\longrightarrow \oplus^2 I_Z(2h_1+3h_2)\longrightarrow I_Z(2h_1+4h_2)\longrightarrow 0,$$

which obviously coincides with the sequence induced by taking direct image of the sequence (#-3) through the morphism ρ.

Hence, it is enough to see that the exactness of the H^0 part of the short sequence (#-5). Now we see that the sheaf O_Z is a skyscraper sheaf $\oplus_{i=1}^7\mathbb{C}(p_i)$ with support on the given 7 points. By applying Lemma 3.5, for an integer $b\geq 2$, if we choose any 6 points in Z, then there is a section $s\in H^0(U,O_U(2h_1+bh_2))$ such that it vanishes on the chosen 6 points in Z and does not vanish on the remaining one point of Z, which brings an exact sequence:

(#-6)
$$0\longrightarrow H^0(I_Z(2h_1+bh_2))\longrightarrow H^0(O_U(2h_1+bh_2))\longrightarrow H^0(O_Z(2h_1+bh_2))\longrightarrow 0.$$

By using $H^1(O_U(2h_1+2h_2))=0$, we have an exact commutative diagram:

(#-7)

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & H^0(O_Z(2h_1+2h_2)) \arrow[r] \arrow[u] & \oplus^2 H^0(O_Z(2h_1+3h_2)) \arrow[r] \arrow[u] & H^0(O_Z(2h_1+4h_2)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^0(O_U(2h_1+2h_2)) \arrow[r] \arrow[u] & \oplus^2 H^0(O_U(2h_1+3h_2)) \arrow[r] \arrow[u] & H^0(O_U(2h_1+4h_2)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^0(I_Z(2h_1+2h_2)) \arrow[r] \arrow[u] & \oplus^2 H^0(I_Z(2h_1+3h_2)) \arrow[r, "\lambda"'] \arrow[u] & H^0(O_Z(2h_1+bh_2)). \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

By Snake lemma, we can conclude the surjectivity of the map λ. ∎

At the end of this section, we make a brief historical remark on the surfaces constructed here and so on.

Remark 3.8 *The surfaces V constructed in Lemma 3.6 are the same as the surfaces called classically as Castelnuovo surfaces (cf. [12]). These surfaces are given as the images of the projective plane $\mathbb{P}^2(\mathbb{C})$ by rational maps, which are composed of blow-ups, non-isomorphic morphisms and projective embeddings. These non-isomorphic morphisms might cause the difficulty in the fine calculation on syzygies. On the other hand, Prof. A. Ohbuchi kindly showed the author that there are trigonal canonical curves on these Castelnuovo surfaces. However, these facts are still far from showing the existence of homological shells of Δ-genera 2, namely to see the global Tor injectivity condition. Thus we need to analyze the non-isomorphic morphisms carefully and show that they are only blow-downs and the images of the morphisms are smooth in our construction. Then we see that it is enough to start not from the projective plane $\mathbb{P}^2(\mathbb{C})$ but from the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The author guesses that Lemma 3.4, Lemma 3.5, and Lemma 3.6 are not new essentially and are known classically for the points chosen generically. But he could not find any appropriate reference which describes the position of points precisely such as "in multi-general position". Since we have to apply Lemma 3.5 for fixed 6 points in the proof of Theorem 3.7, we need a precise description on the position of points as in Definition 3.1. This is the reason why we add three lemmas including proves here.*

§4 Algebraic Construction

To clarify the motivation of this algebraic construction of a homological shell surface of Δ-genus 2, let us review our explicit minimal free resolutions for $g = 5$ trigonal canonical curves which is a little bit of refinement of the ones given in [11]. This is also a typical example for the structure theorem in [3] on the minimal free resolutions in the case of codimension 3 and Gorenstein.

Since the trigonal curve X of genus 5 is contained by a unique rational ruled surface V_0 of degree 3 in $\mathbb{P}^4(\mathbb{C})$ (or equivalently $\Delta(V_0, O_{V_0}(1)) = 0$), we first fix the embedding of the surface V_0 in $\mathbb{P}^4(\mathbb{C}) = P$ as follows.

Taking $B = \mathbb{P}^2(\mathbb{C}) = Proj(\mathbb{C}[U_0, U_1, U_2])$, we get the surface V_0 by blowing up the plane B at the center $p_0 = [1 : 0 : 0]$. The embedding of V_0 by the linear system $|2\xi + E_1|$, namely the linear system coming from the conics passing through the point p_0, is given by $[Z_0 : Z_1 : Z_2 : Z_3 : Z_4] = [U_0U_1 : U_0U_2 : U_1^2 : U_1U_2 : U_2^2]$. The equations of V_0 are : $G_1 = Z_0Z_3 - Z_1Z_2 = 0$, $G_2 = Z_1Z_3 - Z_0Z_4 = 0$, $G_3 = Z_2Z_4 - Z_3^2 = 0$. When we take a 2×3-matrix :

$$\Phi = \begin{bmatrix} Z_4 & Z_3 & Z_1 \\ Z_3 & Z_2 & Z_0 \end{bmatrix}, \tag{\#-8}$$

those equations G_1, G_2, G_3 can be considered as $G_1 = \varphi_{2,3}$, $G_2 = -\varphi_{1,3}$, $G_3 = \varphi_{1,2}$, where the determinant of 2×2-minor matrix made by choosing i, j columns of Φ is denoted by $\varphi_{i,j}$. The curve X arises from the blowing up at the center p_0 of the plane curve C_0 of degree 5 which is smooth outside the point p_0 and has only one double point at the point p_0. An equation $\widehat{F_0}$ of the plane curve C_0 is written by

$$\widehat{F_0} = U_0^3\{a_1U_1^2 + a_2U_1U_2 + a_3U_2^2\} + \sum_{3 \le e_1+e_2 \le 5} c_{e_1,e_2} U_0^{5-e_1-e_2} U_1^{e_1} U_2^{e_2}, \tag{\#-9}$$

where $(a_1, a_2, a_3) \neq (0, 0, 0)$ and the coefficients a_i, c_{e_1,e_2} are constants.

Let us consider the following 3 polynomials $\widehat{F}_{0,1}$, $\widehat{F}_{0,2}$, and $\widehat{F}_{0,3}$.

$$\begin{aligned} \widehat{F}_{0,1} &= c_{0,5}U_2^4 \\ \widehat{F}_{0,2} &= -\sum_{e_1+e_2=5,\ e_1 \ge 1} c_{e_1,e_2} U_1^{e_1-1} U_2^{e_2} \\ \widehat{F}_{0,3} &= U_0^2\{a_1U_1^2 + a_2U_1U_2 + a_3U_2^2\} + \sum_{3 \le e_1+e_2 \le 4} c_{e_1,e_2} U_0^{4-e_1-e_2} U_1^{e_1} U_2^{e_2} \end{aligned} \tag{\#-10}$$

Then we see that $\widehat{F_0} = U_0\widehat{F}_{0,3} - U_1\widehat{F}_{0,2} + U_2\widehat{F}_{0,1}$. From these 3 homogeneous polynomials of degree 4 ; $\widehat{F}_{0,1}$, $\widehat{F}_{0,2}$, $\widehat{F}_{0,3}$, we construct 3 homogeneous polynomials Q_1, Q_2, Q_3 of degree 2 with the variables $Z_0, \cdots Z_4$ by replacing U_0U_1, U_0U_2, U_1^2, U_1U_2, U_2^2 by Z_0, Z_1, Z_2, Z_3, Z_4, respectively. Namely,

$$\begin{aligned} Q_1(Z_4) &= c_{0,5}Z_4^2 \\ Q_2(Z_2, Z_3, Z_4) &= -(c_{5,0}Z_2^2 + c_{4,1}Z_2Z_3 + c_{3,2}Z_3^2 + c_{2,3}Z_3Z_4 + c_{1,4}Z_4^2) \\ Q_3(Z_0, Z_1, Z_2, Z_3, Z_4) &= a_1Z_0^2 + a_2Z_0Z_1 + a_3Z_1^2 \\ &\quad + c_{3,0}Z_0Z_2 + c_{2,1}Z_0Z_3 + c_{1,2}Z_1Z_3 + c_{0,3}Z_1Z_4 \\ &\qquad + c_{4,0}Z_2^2 + c_{3,1}Z_2Z_3 + c_{2,2}Z_3^2 + c_{1,3}Z_3Z_4 + c_{0,4}Z_4^2. \end{aligned} \tag{\#-11}$$

We set 2 homogeneous polynomials F_1 and F_2 of degree 3 with the variables $Z_0, \cdots, Z_4$:

$$\begin{aligned} F_1 &:= Z_3Q_1 - Z_2Q_2 + Z_0Q_3 \\ F_2 &:= -Z_4Q_1 + Z_3Q_2 - Z_1Q_3. \end{aligned} \tag{\#-12}$$

To simplify the notation, we set

$$M := [F_1, F_2], \quad \Psi = \begin{bmatrix} \varphi_{2,3} \\ -\varphi_{1,3} \\ \varphi_{1,2} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & Q_3 & Q_2 \\ -Q_3 & 0 & Q_1 \\ -Q_2 & -Q_1 & 0 \end{bmatrix}. \tag{\#-13}$$

Then a minimal of the homogeneous coordinate ring R_X of the curve X is given explicitly as follows.

(#-14)

$$0 \longleftarrow R_X \longleftarrow S \xleftarrow{\delta_1} S(-2)^{\oplus 3} \oplus S(-3)^{\oplus 2}$$

$$\xleftarrow{\delta_2} S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2} \xleftarrow{\delta_3} S(-6) \longleftarrow 0,$$

where each differential map is defined by a matrix :

$$\begin{aligned} \delta_1 &= [{}^t\Psi, M] = [\varphi_{2,3},\ -\varphi_{1,3},\ \varphi_{1,2},\ F_1,\ F_2] \\ \delta_2 &= \begin{bmatrix} Q & {}^t\Phi \\ -\Phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q_3 & Q_2 & Z_4 & Z_3 \\ -Q_3 & 0 & Q_1 & Z_3 & Z_2 \\ -Q_2 & -Q_1 & 0 & Z_1 & Z_0 \\ -Z_4 & -Z_3 & -Z_1 & 0 & 0 \\ -Z_3 & -Z_2 & -Z_0 & 0 & 0 \end{bmatrix} \\ \delta_3 &= \begin{bmatrix} -\Psi \\ -{}^tM \end{bmatrix} = \begin{bmatrix} -\varphi_{2,3} \\ \varphi_{1,3} \\ -\varphi_{1,2} \\ -F_1 \\ -F_2 \end{bmatrix}. \end{aligned} \tag{\#-15}$$

These matrices act from the left, and every element in a S-free module is considered as a column vector with S-valued coefficients.

Using the resolution above, we can consider the condition that the trigonal canonical curve of genus 5 has a homological shell surface of Δ-genus 2.

Lemma 4.1 (cf. [16]) *Let $X \subset \mathbb{P}^4(\mathbb{C}) = P$ be a trigonal curve of genus 5. Then, in the following three conditions, each former condition implies the latter condition.*

(4.1.1) *There exists a homological shell surface W of X with Δ-genus 2.*

(4.1.2) *In the resolution (#-14), there are two surjective S-linear homomorphisms $p_1 : S(-2)^{\oplus 3} \oplus S(-3)^{\oplus 2} \to S(-2)^{\oplus 2}$ and $p_2 : S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2} \to S(-4) \oplus S(-3)^{\oplus 2}$ such that the homomorphism $p_1 \circ (\delta_2|_K) : K := Ker(p_2) \to S(-2)^{\oplus 2}$ is a zero map.*

(4.1.3) *We can find suitable two 2×2-matrices H, H' with coefficients in homogeneous polynomials of S in degree 1, and 2×3-matrix T, and 3×2 matrix T' whose rank are 2 and both coefficients are constants such that*

$$T \cdot Q \cdot T' = H \cdot \Phi \cdot T' - T \cdot ({}^t\Phi) \cdot H'.$$

To see the relation among the three conditions: (4.1.1) $\sim$ (4.1.3) above, let us recall the exact commutative diagram induced by the strongest condition (4.1.1) :

(#-16)

$$\begin{tikzcd}
 & & & S(-2)^{\oplus 2} & \begin{matrix} S(-4) \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} & & \\
0 & R_X \arrow[l] & S \arrow[l] & \begin{matrix} S(-2)^{\oplus 3} \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta_1"'] \arrow[u, "p_1"] & \begin{matrix} S(-4)^{\oplus 3} \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta_2"'] \arrow[u, "p_2"'] & S(-6) \arrow[l, "\delta_3"'] & 0 \arrow[l] \\
0 & R_W \arrow[l] \arrow[u] & S \arrow[l] \arrow[u, equal] & \begin{matrix} S(-2) \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta'_1"'] \arrow[u, "i_1"] & S(-4)^{\oplus 2} \arrow[l, "\delta'_2"'] \arrow[u, "i_2"'] & 0, \arrow[l] &
\end{tikzcd}$$

where the homomorphisms i_1 and i_2 are induced as a part of an induced homomorphism of chain complexes from the natural ring homomorphism $R_W \to R_X$, and the homomorphisms p_1 and p_2 are defined by canonical quotient maps.

On the other hand, the weakest but the most complicated condition (4.1.3) essentially arises from the zero map $p_1 \circ \delta_2 \circ i_2$ in the diagram (#-16) above. In general situation, it is not so easy to find the matrix T, T', H, H' as in the weakest condition (4.1.3). However, if we consider that the essential point of the condition (4.1.3) is to make a 2×2 zero matrix from the 5×5 matrix of δ_2, the easiest situation is the case that the original 5×5 matrix of δ_2 contains 2×2 zero matrix without any transformation. So we have only to find a good curve such that one of the three quadrics Q_1, Q_2, Q_3 vanishes.

On the other hand, the original plane quintic curve $C_0 = \{\widehat{F}_0 = 0\}$ must satisfy the two conditions: (1) irreducibility ; (2) the smoothness except the point p_0. Reviewing the definition (#-11) of the quadrics Q_1, Q_2, Q_3, the quadric Q_1 is the easiest one to be vanished without losing the controlablility for making the curve C_0 satisfy the two conditions above.

In the affine coordinates $x = U_1/U_0$, $y = U_2/U_0$, the affine equation $\widehat{f}_0(x,y) := \widehat{F}_0/U_0^5$ starts from the term of degree 2 and ends at the term of degree 5 which does not contain the term y^5.

For example, let us consider a quintic equation $\widehat{f}_0(x,y) := xy + y^4 + x^5$. It is very easy to verify the irreducibility of the polynomial $\widehat{f}_0(x,y)$. Its homogenized polynomial $\widehat{F}_0$ has the form :

$$\widehat{F}_0 = U_0^3U_1U_2 + U_0U_2^4 + U_1^5.$$

Then, it is also easy to show that the curve $C_0 = \{\widehat{F}_0 = 0\}$ has only singular point at $p_0 = [1:0:0]$. By the the definition (#-11), we get

$$Q_1 := 0, \quad Q_2 := -Z_2^2, \quad Q_3 := Z_0Z_1 + Z_4^2.$$

Next we apply the definition (#-12), we obtain

$$F_1 := Z_2^3 + Z_0^2Z_1 + Z_0Z_4^2, \quad F_2 := -Z_2^2Z_3 - Z_0Z_1^2 - Z_1Z_4^2.$$

In this case, by the reason that $Q_1 = 0$, for the weakest condition (4.1.3), we have only to set $H = O_{2\times 2}$, $H' = O_{2\times 2}$, $T = [O_{2\times 1}, E_2]$, $T' = {}^t[O_{2\times 1}, E_2]$ where E_2 denotes 2×2 identity matrix, and $O_{p\times q}$ does the zero matrix of type $p \times q$.

To construct the homological shell surface W of Δ-genus 2, observing the diagram (#-16), we have to choose suitable two rows from the 5×5 matrix δ_2 for the matrix δ'_2. Then we can find that the 3×2 matrix δ'_2 should be constructed from the second row and the third row of the 5×5 matrix δ_2, namely

$$\delta'_2 = \begin{bmatrix} Q_3 & Q_2 \\ -Z_3 & -Z_1 \\ -Z_2 & -Z_0 \end{bmatrix} = \begin{bmatrix} Z_0Z_1 + Z_4^2 & -Z_2^2 \\ -Z_3 & -Z_1 \\ -Z_2 & -Z_0 \end{bmatrix} \tag{\#-17}$$

Then, for the 1×3 matrix δ'_1, it is natural to choose the first row, the fourth row, and the fifth row of the 1×5 matrix δ_1, namely

(#-18)
$$\delta'_1 = [G_1 = \varphi_{2,3}, F_1, F_2] = [Z_0Z_3 - Z_1Z_2,\ Z_2^3 + Z_0^2Z_1 + Z_0Z_4^2,\ -Z_2^2Z_3 - Z_0Z_1^2 - Z_1Z_4^2].$$

Thus our candidate for a homological shell surface W of Δ-genus 2 for the curve X is defined by the system of equations:

$$\begin{cases} Z_0Z_3 - Z_1Z_2 = 0 \\ Z_2^3 + Z_0^2Z_1 + Z_0Z_4^2 = 0 \\ -Z_2^2Z_3 - Z_0Z_1^2 - Z_1Z_4^2 = 0. \end{cases} \tag{\#-19}$$

Then it is easy to check that the sequence:

$$0 \longleftarrow R_W \longleftarrow S \xleftarrow{\delta'_1} \begin{matrix} S(-2) \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \xleftarrow{\delta'_2} S(-4)^{\oplus 2} \longleftarrow 0, \tag{\#-20}$$

forms a chain complex, where the matrices δ'_1 and δ'_2 are given by (#-18) and (#-17) above. Moreover, by a rather hard and brutal calculation, we can also check the exactness of the complex (#-20).

Thus we obtain a homological shell surface W of Δ-genus 2 for the curve X, where the canonical curve X is defined by the system of equations

$$\begin{cases} Z_0Z_3 - Z_1Z_2 = 0 \\ Z_1Z_3 - Z_0Z_4 = 0 \\ Z_2Z_4 - Z_3^2 = 0 \\ Z_2^3 + Z_0^2Z_1 + Z_0Z_4^2 = 0 \\ -Z_2^2Z_3 - Z_0Z_1^2 - Z_1Z_4^2 = 0. \end{cases} \tag{\#-21}$$

Remark 4.2 *The surface W defined by the equations (#-19) has only one singularity at the point $q_0 := [0 : 0 : 0 : 1 : 0]$. Since the point q_0 does not satisfy one of the equations (#-21) of the curve X : $Z_2Z_4 - Z_3^2 = 0$, we see that the curve X does not go through the point q_0. Thus, this homological shell surface W is actually a geometric shell of the canonical curve X.*

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