# Two kinds of shell equivalences in the inclusion families of a canonical curve of genus 5

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**Abstract**

By using our previous classification on the homological shells of a canonical curve $X$ of genus 5, we study precisely the parameter spaces of the homological shells of the curve $X$. For that purpose, we introduce newly two kinds of equivalence relations on the homological shells, which are called *strict shell equivalence* and *weak shell equivalence*, respectively.
**Keywords**: homological shell(=pregeometric shell), canonical curve, genus 5, inclusion family, strict shell equivalence, weak shell equivalence, Koszul domain, Koszul graph map(=$\gamma$-map)

## §0 Introduction.

Let us take a projective subvariety (or subscheme) $X \subseteq \mathbb{P}^N(\mathbb{C})$ whose arithmetic depth is greater than or equal to 2. Then, the graded Betti numbers of the homological shells of $X$ are bounded by those of $X$, which implies that the homological shells of $X$ are always bounded, namely move in a finite union of connected algebraic families without fixing their dimensions or their degrees in advance. The assumption above on the arithmetic depth is needed to characterize the locus of Tor-injectivity holding by using a sheaf theoretic method. From this general principle, we can expect the classification of all the homological shells of $X$ and the explicit determination of their algebraic families.

However, when a projective subvariety $X \subseteq \mathbb{P}^N(\mathbb{C})$ with the assumption above is given concretely, it is not so easy in general to carry out this classification or even to determine the tangent spaces of the parameter spaces of its homological shells since it needs to determine explicitly the supports of the certain (higher) direct image sheaves and those of the cokernel sheaves of the induced homomorphisms for their pairs on the Hilbert schemes, which are still unidentified mostly (cf. [**?**]).

On the homological shells of a canonical curve $X \subseteq \mathbb{P}^4(\mathbb{C})$ of genus 5 (cf. [**?**], [**?**] [**?**]), by applying the theory of $\Delta$-genus (cf. [**?**]), we can analyze closed subschemes of $\mathbb{P}^4(\mathbb{C})$ precisely which are candidates of the homological shells of $X$ after classifying their Hilbert polynomials. Then it brings us the complete classification on the homological shells of the curve $X$ as in the inclusion diagrams in §2 (see also §4 of [**?**]). Since there are lots of classical researches on the canonical curves of genus 5 itself (e.g. cf. [**?**], [**?**], [**?**], [**?**], [**?**]), here we have a chance to carry out the explicit determination of the parameter spaces for algebraic families of its homological shells without analyzing the higher direct image sheaves and so on. In general, the movements of all the homological shells of the given projective variety $X$ with preserving all their inclusions are similar to, or more complicated than the movements of the full flags in a linear space. Thus we study only the parameter spaces of the homological shells of the curve $X$ with preserving

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a one step inclusion. For that purpose, we introduce newly two kinds of equivalence relations on the homological shells, which are called *strict shell equivalence* and *weak shell equivalence*, respectively.

In our old work [**?**], we generally established the one to one correspondence between the homological shells of codimension 1 up to a suitable equivalence relation and the obstruction classes for a certain kind of infinitesimal liftings (cf. Example 1.9). This equivalence relation for the codimension 1 homological shells is generalized to the strict shell equivalence for the homological shells of higher codimensions. Thus we can expect that the strict shell equivalence will play an important role in our future study on the homological shells of higher codimensions with modeling after Galois theory. On the other hand, similar to the equivalence up to deformation, the weak shell equivalence describes the natural continuous movement, or the embedded deformation of the homological shells with preserving their graded Betti numbers.

In this article, we use successively the notation and conventions in [**?**], [**?**], [**?**], [**?**], [**?**], and [**?**] without mention.

# §1 Preliminaries.

Before we introduce two key concepts on the equivalence of homological shells, we give several definitions and their remarks.

**Definition 1.1 (graded Betti numbers)** *Take a projective scheme $X \subseteq \mathbb{P}^N = P = Proj(S)$ and the homogeneous coordinate ring $R_X = S/\mathbb{I}_X$ of $X$, where $S = \mathbb{C}[Z_0, \ldots, Z_N]$ and $\mathbb{I}_X$ is the homogeneous ideal which defines $X$ and does not have the irrelevant maximal ideal $S_+ = (Z_0, \ldots, Z_N)S$ as an associated homogeneous prime ideal. We denote the $i$-th Betti number in degree $j$ of $X$ by $\beta_{i,j}(X)$ or simply by $\beta_{i,j} = \beta(i,j)$, namely $\beta_{i,j}(X) = \dim_{\mathbb{C}} Tor_i^S(R_X, S/S_+)_{(j)}$. In other words, the graded ring $R_X$ has a graded minimal $S$-free resolution $\mathbb{F}_{X,\bullet}$ of the form $\mathbb{F}_{X,i} = \oplus S(-j)^{\oplus\beta(i,j)}$.*

**Remark 1.2** (i) *In Definition* **??***, if we take a homological shell $W$ of $X$, then we have $\beta_{i,j}(X) \geq \beta_{i,j}(W)$ for any $i$ and $j$. Moreover, if we have another homological shell $W'$ of $X$ with $W \subseteq W'$, then we see that $W'$ is also a homological shell of $W$ and $\beta_{i,j}(W) \geq \beta_{i,j}(W')$ for any $i$ and $j$.* (ii) *If the graded ring $R_X$ satisfies the arithmetic $D_2$ condition, namely has the arithmetic depth being greater than or equal to 2, or equivalently $R_X = \oplus_m H^0(X, O_X(m))$, then $\beta_{i,j}(X) = 0$ for $i = N, N+1$. The converse is also true.*

**Definition 1.3 (a family of homological shells)** *Let $V \subseteq \mathbb{P}^N(\mathbb{C}) = P$ and $\mathcal{W} \subseteq P \times B$ be closed subschemes, where $B$ denotes a connected algebraic scheme over $\mathbb{C}$. Assume that*

(i) *$V \times B \subseteq \mathcal{W}$ ;*

(ii) *the second projection $pr_2 : P \times B \to B$ induces a flat morphism $f = pr_2|_{\mathcal{W}} : \mathcal{W} \to B$;*

(iii) *for any closed point $b \in B$, the fiber $W_b = W(b) = f^{-1}(b) \subseteq P \times k(b) \cong P$ is a homological shell of $V \times k(b) \cong V$ ;*

(iv) *there are the constants $\{c_{i,j}\}$ satisfying $0 \leq c_{i,j} \leq \beta_{i,j}(V)$ $(\forall i, j)$ and each of the graded Betti numbers satisfies $\beta_{i,j}(W_b) = c_{i,j}$ for any closed point $b \in B$ and for any $i$, $j$, respectively.*

*Then we say that the family $f : \mathcal{W} \to B$ is* a weak family of homological shells *of $V$ with the type $\{c_{i,j}\}$, or simply* a family of homological shells *of $V$, and that $B$ is* the parameter space *of the family.*

*Taking a non-negative integer $k = 0, 1, \ldots, h(V) = hd_S(R_V)$, if we assume moreover*

($\mathrm{v}_k$) *the subspace $T_{i,j}(W(b)) := Tor_i^S(R_{W(b)}, S/S_+)_{(j)}$ defined by an induced homomorphism of Tor-groups from the canonical map $R_{W(b)} \to R_V$ does not move in the space $T_{i,j}(V) := Tor_i^S(R_V, S/S_+)_{(j)}$ for any $i \leq k$ and $j$,* *then we say* $f : \mathcal{W} \to B$ *is* a $k$-weak family of homological shells *of* $V$. *In case of* $k = h(V) = hd_S(R_V)$, *namely a* $h(V)$*-weak family of homological shells is also called as* a strict family of homological shells *of* $V$. *Obviously,* $0$*-weak family of homological shells of* $V$ *is the same concept as (weak) family of homological shells of* $V$.

In the situation of Definition **??**, once we have a family $f : \mathcal{W} \to B$ of homological shells of $V$ with the type $\{c_{i,j}\}$, then, by setting $r(V) := \max\{j | T_{i,j}(V) \neq 0\}$, we obtain a natural morphism

$$\gamma(f) : B \to G := G(\{c_{i,j}\}, V) = \prod_{\substack{0 \le i \le h(V) \\ 0 \le j \le r(V)}} Grass(c_{i,j}, T_{i,j}(V)),$$

which sends a closed point $b \in B$ to the point $\prod T_{i,j}(W(b))$. Here $Grass(c, T)$ denotes the Grassmannian variety which parametrizes all the subspaces of dimension $c$ in the vector space $T$. This is an analogue of the period map induced from a variation of the Hodge structures. Taking a non-negative integer $k = 0, 1, \ldots, h(V) = hd_S(R_V)$, we can consider a partial product space :

$$G(k) := G(k;\ \{c_{i,j}\}, V) = \prod_{\substack{0 \le i \le k \\ 0 \le j \le r(V)}} Grass(c_{i,j}, T_{i,j}(V)),$$

a natural projection morphism $p_k : G \to G(k)$, and a composite morphism $\gamma(f)_k = p_k \circ \gamma(f) : B \to G(k)$. Then, for any closed point $t \in G(k)$, taking a connected component $B_t^\circ$ of the fiber $B_t := \gamma(f)_k^{-1}(t)$, the family $f_t^\circ : \mathcal{W}_t^\circ := \mathcal{W} \times_B B_t^\circ \to B_t^\circ$ forms a $k$-weak family of homological shells of $V$.

**Definition 1.4 (Koszul domain, Koszul graph map)** *The target spaces* $G$ *and* $G(k)$ *above are called as* the Koszul domain *of* $V$ *with the type* $\{c_{i,j}\}$ *and* $k$-part of the Koszul domain $G$, *respectively. The induced morphisms* $\gamma(f)$ *and* $\gamma(f)_k$ *above are called as* the Koszul graph map (or simply $\gamma$-map) *of* $f$ *and* $k$-th part of the Koszul graph map (or simply $\gamma_k$-map), *respectively. Moreover, we set*

$$G(k)^\perp := G(k;\ \{c_{i,j}\}, V)^\perp = \prod_{\substack{k+1 \le i \le h(V) \\ 0 \le j \le r(V)}} Grass(c_{i,j}, T_{i,j}(V)),$$

*which is the cofactor of* $G(k)$ *in* $G$ *and is called as the* $k$*-copart of the Koszul domain* $G$. *Composing a projection* $G \to G(k)^\perp$ *with the* $\gamma$*-map* $\gamma(f)$, *we have a morphism* $\gamma(f)_k^\perp : B \to G(k)^\perp$, *which is called as* the $k$-th copart of the $\gamma$-map $\gamma(f)$. *Similarly, from the projection* $G \to Grass(c_{i,j}, T_{i,j}(V))$, *we can construct a morphism* $\gamma(f)_{(i,j)} : B \to Grass(c_{i,j}, T_{i,j}(V))$, *which is called as the* $(i, j)$*-part of* $\gamma$*-map.*

The idea described above can be also generalized easily for handling the families of *homological shell chains* of the full length (cf. [**?**]) in the (strict) inclusion diagram (cf. §2) with replacing the Grassmannian varieties in the target space by Flag varieties. On the other hand, in a strict inclusion diagram, if we take two homological shell chains in the full length, they may have several junction points or diverging points. Thus, if we are interested in the whole inclusion diagram, we might have to construct a new target space which reflects the graph topology of the strict inclusion diagram by introducing incidence relations on the several products of the target spaces for the homological chains in the full length. Anyway, it is very hard for us to construct in general the (strict) inclusion diagram for a given projective subvariety $V$, we have to leave this task with the researches on the homological shells in future.

**Remark 1.5 (Flatness)** *In the situation of Definition* **??***, if the base scheme $B$ is* reduced*, then the condition* (iv) *implies automatically the flatness condition* (ii)*. Thus, if we have a family $\mathcal{W} \to B$ with the three conditions* (i), (iii)*, and* (iv) *but without the condition* (ii)*, then the family $\mathcal{W}\,' := \mathcal{W} \times_B B_{red} \to B_{red}$ satisfies all the four conditions* (i)-(iv) *and becomes a weak family of homological shells of $V$.*

*To see that the flatness condition* (ii) *is implied by the condition* (iv)*, since the Hilbert polynomial of a fiber $W_b = f^{-1}(b)$ over a closed point $b \in B$ is determined by its graded Betti numbers $\{\beta_{i,j}(W_b)\}$ which are independent from the choice of the closed point $b \in B$ by the condition* (iv)*, we have only to modify slightly Theorem 9.9 in Chap.*III *of* [**?**] *as follows.*

> **Modified Theorem 9.9** *Let $T$ and $X \subseteq \mathbb{P}^n(\mathbb{C}) \times T$ be a reduced algebraic scheme over $\mathbb{C}$ and a closed subscheme, respectively. For each closed point $t \in T$, we consider the Hilbert polynomial $H_t(m) \in \mathbb{Q}[m]$ of the fiber $X_t$ considered as a closed subscheme of $\mathbb{P}^n(\mathbb{C}) \cong \mathbb{P}^n(\mathbb{C}) \times k(t)$. Then $X$ is flat over $T$ if and only if the Hilbert polynomial $H_t(m)$ is independent of $t$.*

*To show the modified Theorem 9.9, we have only to replace the freeness of the modules in the proof of the original Theorem 9.9 of* [**?**] *by the locally freeness of the modules, and to use the fact that the scheme $T$ has enough closed points. For example, Lemma 8.9 in Chap.*II *of* [**?**] *should be read that if the ring $A$ is a reduced ring of finite type over $\mathbb{C}$ and the module $M$ is a finite $A$-module which satisfies the constancy of* $\dim_{k(b)} M \otimes k(b)$ *for all the closed point $b \in Spec(A)$, then the module $M$ is locally free.*

*It is well-known that the flatness does not imply the constancy of the numbers of equations, or more generally that of the graded Betti numbers. In other words, the concept of flatness is too weak for handling the variations of syzygies. Thus, we can say that our definition of* (*weak or strict*) *families of homological shells is still in the stage of ad hoc, and can not extract the full power of the base scheme $B$* (cf. [**?**]).

Now let us give our two key concepts, namely the definition of two equivalence relations on homological shells.

**Definition 1.6 (weak or strict shell equivalence)** *Let $V \subseteq \mathbb{P}^N(\mathbb{C}) = P$ be a closed subscheme. Take two homological shells $W$, $W' \subseteq P$ of $V$ and a non-negative integer $k = 0, 1, \ldots, h(V) = hd_S(R_V)$. We say that the homological shells $W$ and $W'$ are $k$-*weakly shell equivalent *with each other and denote $W \underset{w(k)}{\sim} W'$ if there exist finite number of homological shells $W_0, \ldots W_\ell$ with $W = W_0$ and $W' = W_\ell$ of $V$ and $k$-weak families $f_a : \mathcal{W}_a \to B_a$ of homological shells of $V$ $(a = 1, \ldots, \ell)$ and closed points $b_{a,0}$, $b_{a,1} \in B_a$ $(a = 1, \ldots, \ell)$ which satisfy $W_{a-1} = f_a^{-1}(b_{a,0})$ and $W_a = f_a^{-1}(b_{a,1})$ for all $a$. We simply call $0$-weak shell equivalence and $h(V)$-weak shell equivalence as* weak shell equivalence *and* strict shell equivalence*, respectively. In this case, we usually use the symbols $W_0 \underset{w}{\sim} W_1$ and $W_0 \underset{s}{\sim} W_1$ instead of $W_0 \underset{w(0)}{\sim} W_1$ and $W_0 \underset{w(h(V))}{\sim} W_1$, respectively.*

Under the same circumstances, let us discuss roughly the families of homological shells of $V$ from the view point of the Hilbert schemes (for more precise discussion, see [**?**]). Let us take a closed subscheme $V \subseteq \mathbb{P}^N(\mathbb{C}) = P$ which satisfies arithmetic $D_2$-condition, namely the condition that the depth of the local ring at the vertex of the affine cone of $V$ is greater than or equal to 2. Next we take a homological shell $W_0$ of $V$ and consider any homological shell $W$ which is weakly shell equivalent to $W_0$. Then, by the definition, we see that the homological shell $W$ satisfies also arithmetically $D_2$-condition and its graded Betti numbers satisfy $\beta_{i,j}(W) = \beta_{i,j}(W_0)$ $(\forall i = 0, 1, \ldots h(V), j = 0, 1, \ldots r(V))$, which implies the coincidence of their Hilbert Polynomials $A_W(m) = A_{W_0}(m)$. Let us set $c_{i,j} = \beta_{i,j}(W_0)$ and $A(m) = A_{W_0}(m)$. Then we consider the Hilbert scheme $Hilb_P^{A(m)}$, which is well-known to be a projective scheme and to have the universal family $\pi_0 : \mathcal{U}_0 \to Hilb_P^{A(m)}$. Now we set $H_0 = Hilb_P^{A(m)}$ and take a subset $H(A(m), V, D_2) \subseteq H_0$

parametrizing all the closed schemes over $H_0$ which includes the scheme $V$ and satisfies the arithmetic $D_2$-condition. It is easy to show that the set $H_1 = H(A(m), V, D_2)$ is a locally closed set in the sense of Zariski topology. Attaching the reduced scheme structure on $H_1$ from the scheme structure of $H_0$, we have a reduced quasiprojective scheme $H_1$ and a family $\pi_1 : \mathcal{U}_1 = \mathcal{U}_0 \times_{H_0} H_1 \to H_1$ by taking a natural fiber product. Next we take a subset $H_2 = H(V, D_2, \{c_{i,j}\})$ parametrizing a closed scheme $W'$ over $H_1$ whose graded Betti numbers satisfy $\beta(W')_{i,j} = c_{i,j}$ $(\forall i = 0, 1, \dots h(V), j = 0, 1, \dots r(V))$. Then it is also easy to show that the set $H_2$ is a locally closed set of the scheme $H_1$ since the upper semi continuity of the graded Betti numbers holds on the algebraic family $\pi_1 : \mathcal{U}_1 \to H_1$ (cf. not on the whole family $\pi_0 : \mathcal{U}_0 \to H_0$). By the same argument above, we have a reduced quasiprojective scheme $H_2$ and a family $\pi_2 : \mathcal{U}_2 = \mathcal{U}_1 \times_{H_1} H_2 \to H_2$. Now we consider the set $\widetilde{B} = HS(\{c_{i,j}\}, V)$ which parametrizes a homological shell $W$ of $V$ with the condition: $\beta(W)_{i,j} = c_{i,j}$ $(\forall i = 0, 1, \dots h(V), j = 0, 1, \dots r(V))$. Obviously $HS(\{c_{i,j}\}, V)$ is a subset of a quasiprojective scheme $H_2$. After a precise argument with using higher direct image sheaves, we can see the set $\widetilde{B} = HS(\{c_{i,j}\}, V)$ is also a locally closed set of $H_2$. By attaching the reduced structure on the scheme $HS(\{c_{i,j}\}, V)$ and taking a fiber product, we have a family:

$$\widetilde{f} : \widetilde{\mathcal{W}} = \mathcal{U}_2 \times_{H_2} \widetilde{B} \longrightarrow \widetilde{B} = HS(\{c_{i,j}\}, V),$$

which induces the morphisms:

$$\gamma(\widetilde{f})_k : HS(\{c_{i,j}\}, V) \longrightarrow G(k;\ \{c_{i,j}\}, V)$$

.

for $k = 0, 1, \dots, h(V)$.

**Definition 1.7 (Inclusion Family)** *The quasiprojective scheme $HS(\{c_{i,j}\}, V)$ or the family $\widetilde{f} : \widetilde{\mathcal{W}} \to HS(\{c_{i,j}\}, V)$ obtained above is called* an inclusion family *of $V$ with the type $\{c_{i,j}\}$. Once we obtain a strict inclusion diagram, we often choose a representative $W_0 \in HS(\{c_{i,j}\}, V)$ arbitrarily and denote simply $\alpha : V \to W_0$. Then the parameter spaces of the weak shell equivalences and the strict shell equivalences are described by $B^w(\alpha)$ and $B^s(\alpha)$, respectively. Or more simply, we denote $B^w(\alpha)/B^s(\alpha)$. Speaking more precisely, set the quotient set $\Delta = HS(\{c_{i,j}\}, V)/ \underset{w}{\sim}$ by the weak shell equivalence, and choose representatives $W_\delta \in HS(\{c_{i,j}\}, V)$ $(\delta \in \Delta$ and $\delta = [W_\delta]$ )Then, $HS(\{c_{i,j}\}, V) = B^w(\alpha) = \coprod_\delta B^w(\alpha)_\delta$ and $B^w(\alpha)_\delta = \{W \in HS(\{c_{i,j}\}, V) | W \underset{w}{\sim} W_\delta\}$. In case of $\#\Delta = 1$, namely $\Delta = \{1pt\} = \{\delta_0\}$, then we simply denote $B^w(\alpha) = B^w(\alpha)_{\delta_0}$. On the other hand, for any two distinct elements $\delta_1, \delta_2 \in \Delta$, if we always have an isomorphism $B^w(\alpha)_{\delta_1} \cong B^w(\alpha)_{\delta_2} \cong T$ as abstract schemes, we denote simply $B^w(\alpha) = \sqcup T$. We apply the similar abbreviation to $B^s(\alpha)$ also. We should make a remark that as sets $B^w(\alpha) = B^s(\alpha)$, but their topologies are different in general.*

**Remark 1.8** *In general, the family $\widetilde{f} : \widetilde{\mathcal{W}} \to HS(\{c_{i,j}\}, V)$ does not have the universality for all the families of homological shells of $V$ with the type $\{c_{i,j}\}$ but have the universality only for those whose parameter spaces are reduced.*

Let us see the relation between the strict shell equivalence and our previous works (cf. [**?**], [**?**], [**?**]). Since our terminology has changed at several times by the progress of our research, we summarize the relation by using the latest terminology.

**Example 1.9** *Let $V \subseteq \mathbb{P}^N(\mathbb{C}) = P$be a projective integral scheme with arithmetic $D_2$-condition. Consider the infinitesimal lifting problems of homomorphisms $O_V(-m) \to N^\vee_{V/P} \subseteq \Omega^1_P \otimes O_V$ to the 1-st* *infinitesimal neighborhood $V_{(1)} = (|V|, O_P/I_V^2)$. Their obstruction classes form a subspace of $H^1(V, \Omega_P^1 \otimes N_{V/P}^{\vee}(m))$, which have a one to one correspondence with the homological shells of $V$ with codimension $1$ in $P$, up to a suitable equivalence relation described in the sequel. Since the scheme $V$ is integral, its codimension $1$ homological shell $W$ is always an integral hypersurface of $P$, which has a unique irreducible homogeneous equation $F_W \in \mathbb{I}_V$ up to $\mathbb{C}^*$-multiplication. Then, the equivalence relation mentioned above is given by : two homological shells $W_0$ and $W_1$ of $V$ with codimension $1$ in $P$ are equivalent if and only if their homogeneous equations $F_{W_0}$ and $F_{W_1}$ define the same non-zero class in $\mathbb{I}_V/(S_+ \cdot \mathbb{I}_V) \cong Tor_1^S(R_V, S/S_+)$, namely $F_{W_0}$, $F_{W_1} \notin (S_+ \cdot \mathbb{I}_V)$ and $F_{W_1} = F_{W_0} + G$ with $G \in (S_+ \cdot \mathbb{I}_V)$. Then, by setting $B = Spec(\mathbb{C}[t])$, $\mathcal{W} = Proj(\mathbb{C}[t][Z_0, \ldots, Z_N]/(F_{W_0} + t \cdot G)) \subseteq \mathbb{P}^N \times B$, we have a strict family of homological shells : $f = pr_2|_{\mathcal{W}} : \mathcal{W} \to B$. Since we have $W_0 = f^{-1}(0)$ and $W_1 = f^{-1}(1)$ for $0, 1 \in B \cong \mathbb{A}^1$, we see that $W_0$ and $W_1$ are strictly shell equivalent with each other.*

The next easy example shows the typical difference between the strict shell equivalence and the weak shell equivalence.

**Example 1.10** *Let $V \subseteq \mathbb{P}^N(\mathbb{C}) = P$ $(N \geq 4)$ be a $(2,3,3)$-complete intersection with the homogeneous ideal $\mathbb{I}_V = (G, F_0, F_1)S$ where $\deg F_i = 3$ $(i = 0, 1)$ and $\deg G = 2$. Now we consider a hypersurface $W(\alpha_0 : \alpha_1, L) = \{\alpha_0 \cdot F_0 + \alpha_1 \cdot F_1 + L \cdot G = 0\}$ $([\alpha_0 : \alpha_1] \in \mathbb{P}^1(\mathbb{C})$, $L \in S_1$ (a linear form)). For any $[\alpha_0 : \alpha_1]$, $[\alpha_0' : \alpha_1'] \in \mathbb{P}^1(\mathbb{C})$, $L$, $L' \in S_1$, two hypersurfaces $W(\alpha_0 : \alpha_1, L)$ and $W(\alpha_0' : \alpha_1', L')$ are weakly shell equivalent with each other. On the other hand, two hypersurfaces $W(\alpha_0 : \alpha_1, L)$ and $W(\alpha_0' : \alpha_1', L')$ are strictly shell equivalent with each other if and only if the equality : $[\alpha_0 : \alpha_1] = [\alpha_0' : \alpha_1']$ in $\mathbb{P}^1(\mathbb{C})$ holds.*

*Every homological shell $W$ of $V$ with the codimension $1$ in $P$ and the degree $3$ is written in the form $W(\alpha_0 : \alpha_1, L)$. Hence, if $N = 4$, then the parameter spaces of strict shell equivalence and of weak shell equivalence are $\mathbb{A}^5$ and $\mathbb{P}^1 \times \mathbb{A}^5$, respectively.*

# §2 Main Results.

Let us take a canonical curve $X \subseteq \mathbb{P}^4(\mathbb{C}) = P$ of genus 5. We summarize our results in the following two strict inclusion diagrams, where the diagram in the left hand side is for the case that the curve $X$ is generic, namely of non-trigonal and the one in the right hand side is for the case that the curve $X$ is of trigonal.

$$
\begin{tikzcd}
(X = Q_1 \cap Q_2 \cap Q_3, \Delta = 4) \arrow[d, "\alpha_1"] \\
(Q_1 \cap Q_2, \Delta = 1) \arrow[d, "\alpha_2"] \\
(Q_1, \Delta = 0) \arrow[d, "\alpha_3"] \\
(P, \Delta = 0)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (X = Y \cap Z, \Delta = 4) \arrow[dl, "\alpha_4"'] \arrow[dr, "\alpha_5"] & \\
(Z, \Delta = 2) \arrow[d, "\alpha_6"'] \arrow[drr, "\alpha_7"] & & (Y = \mathbb{F}_1, \Delta = 0) \arrow[d, "\alpha_8"] \\
(D, \Delta = 1) \arrow[dr, "\alpha_9"'] & & (Q, \Delta = 0) \arrow[dl, "\alpha_{10}"] \\
& (P, \Delta = 0) &
\end{tikzcd}
$$

In the diagrams above, all the homological shells are integral. The symbols $\Delta$, $Q_i$, and $D$ denote the $\Delta$-genus, a quadric hypersurface, and a cubic hypersurface, respectively. The varieties $Y \subseteq Q$ and $Z \subseteq Q$ correspond to Cartier divisors $2H - R$ and $2H + R$, respectively, in a rational scroll $\widetilde{S}(1,1,0)$ which is a desingularization of a quadric hypersurface $Q$ of rank 4. The divisors $H$ and $R$ denote the pullback of the hyperplane and the fiber of the scroll, respectively.

The list of the parameter spaces of the weak shell equivalences and of the strict shell equivalences for one step inclusions $\alpha_i$ $(i = 1, \ldots, 10)$ in the strict inclusion diagrams above is given as follows. The symbol $\bullet$ denotes that the parameter space is the one point.

$$B(\alpha_1) = \mathbb{P}^2/\sqcup \bullet \quad B(\alpha_2) = \mathbb{P}^1/\sqcup \bullet \quad B(\alpha_3) = \bullet/\bullet$$

$$B(\alpha_4) = B_4^w/B_4^s \quad B(\alpha_5) = \bullet/\bullet \quad B(\alpha_6) = (\mathbb{P}^1 \times \mathbb{A}^5)/\sqcup \mathbb{A}^5$$

$$B(\alpha_7) = \bullet/\bullet \quad B(\alpha_8) = \mathbb{P}^2/\sqcup \bullet \quad B(\alpha_9) = \bullet/\bullet \quad B(\alpha_{10}) = \bullet/\bullet$$

The whole parameter space $B_4^w$ is connected and coincides with a $\mathbb{P}^2$-fiber space over a quasiprojective reducible curve, or more precisely an open set $D_0$ of a union of 3 lines in $\mathbb{P}^2$ as in Figure **??**.

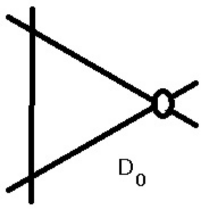


Figure 1: curve $D_0 \subseteq \mathbb{P}^2$

On the other hand, what we know on the parameter spaces $B_4^s$ of the strict shell equivalences is only the fact that there is a morphism $\mu : \mathbb{P}^2 \to \mathbb{P}^2$, and each connected parameter space in $B_4^s$ is contained in the fiber of the morphism $\mu$, which implies that $B_4^s = \mathbb{P}^2$ or each connected parameter space of strict shell equivalence is one point.

# §3 Proof of the Results.

Only in this section, to avoid confusion, we denote the Koszul domain by $\mathbb{G}$ instead of the usual notation $G$ since the quadric equations are denote by using $G$ and so on.

Except $B(\alpha_4)$, by using the similar argument in Example **??**, all the parameter spaces $B(\alpha_i)$ are easily identified as in the list above. Let us concentrate our attention only on the parameter space $B(\alpha_4)$. Thus we take a canonical curve $X \subseteq \mathbb{P}^4(\mathbb{C}) = P$ of genus 5 which is of trigonal. Then, as an inclusion family, we have

$$B(\alpha_4) = HS(c_{0,0} = 1, c_{1,2} = 1, c_{1,3} = 2, c_{2,4} = 2, X)$$

by the results of [**?**]. Now we apply Theorem 2.4 of [**?**] and the result of §4 in [**?**] and see that the whole parameter space $B = B(\alpha_4)$ is a locally closed set of $H_0 = Hilb_P^{A(m)}$ where $A(m) = \frac{5}{2}m^2 + \frac{3}{2}m + 1$. Attaching the reduced structure on $B$ and taking a fiber product, we have a family $f : \mathcal{W} = \mathcal{U} \times_{H_0} B \to B$

and the $\gamma$-map $\gamma(f) : B \to \mathbb{G}$. If $c_{i,j} = 0$ or $c_{i,j} = \beta_{i,j}(X)$, then $Grass(c_{i,j}T_{i,j}(X)) = \bullet$, which means that we can ignore such a factor of the Koszul domain $\mathbb{G}$. Hence we may consider that $\mathbb{G}(2) = \mathbb{G}$, namely

$$\mathbb{G}(2;\ c_{0,0} = 1, c_{1,2} = 1, c_{1,3} = 2, c_{2,4} = 2, X) = \mathbb{G}(c_{0,0} = 1, c_{1,2} = 1, c_{1,3} = 2, c_{2,4} = 2, X).$$

Moreover, since $\beta_{0,0}(X) = 1$, $\beta_{1,2}(X) = 3$, $\beta_{1,3}(X) = 2$, and $\beta_{2,4}(X) = 3$, we see that $\mathbb{G} = Grass(c_{1,2} = 1, \mathbb{C}^3) \times Grass(c_{2,4} = 2, \mathbb{C}^3) \cong \mathbb{P}^2_{(1,2)} \times \mathbb{P}^2_{(2,4)}$, where the subscript $(i,j)$ of $\mathbb{P}^2_{(i,j)}$ is attached for distinguishing the component of the target space. Thus our $\gamma$-map $\gamma(f) : B \to \mathbb{G}$ can be decomposed into $\gamma(f) = \gamma(f)_{(1,2)} \times \gamma(f)_{(2,4)}$, which means that $\gamma(f)_1 = \gamma(f)_{(1,2)}$ and $\gamma(f)_1^\perp = \gamma(f)_{(2,4)}$.

Let us consider the morphism $\gamma(f)_1 = \gamma(f)_{(1,2)} : B \to \mathbb{P}^2_{(1,2)}$ more precisely from the view point of Geometry. Recalling classical results (cf. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]), the quadric hull $Y$ of the trigonal canonical curve $X \subseteq \mathbb{P}^4 = P$ is uniquely determined and isomorphic to the Hirzebruch surface $\mathbb{F}_1$, namely 1-point blow up of $\mathbb{P}^2$. After suitable change of homogeneous coordinates $\mathbb{P}^4 = P$, the surface $Y$ is defined by the following three quadric equations $\{G_0, G_1, G_2\}$.

$$\begin{cases} G_0 = & Z_0Z_3 - Z_1Z_2 = 0 \\ G_1 = & Z_1Z_3 - Z_0Z_4 = 0 \\ G_2 = & Z_2Z_4 - Z_3^2 = 0 \end{cases} \qquad (\#\text{-}1)$$

By the definition of the surface $Y$, the three quadric equations $\{G_0, G_1, G_2\}$ span the space $T_{1,2}(X)$. Hence the homogeneous coordinates $[\tau_0 : \tau_1 : \tau_2] \in \mathbb{P}^2_{(1,2)}$ can be considered as the quadric equation $G = \tau_0 G_0 + \tau_1 G_1 + \tau_2 G_2$. Let us take a homological shell $Z \in B$. As we saw in [**?**] and [**?**], we have $X = Y \cap Z$. On the other hand, $\mathbb{C}^1 \cong T_{1,2}(Z) \subseteq T_{1,2}(X) = T_{1,2}(Y) \cong \mathbb{C}^3$, which shows that there exists a unique quadric equation $\widehat{G}_Z \in \mathbb{I}_Z \cap \mathbb{I}_Y$ up to $\mathbb{C}^*$-multiplication. Then the rank of quadric equation $\widehat{G}_Z$ must be 4, namely a (normal) rational scroll 3-fold with only one singular point : $\widehat{Q} = \{\widehat{G}_Z = 0\} \cong S(1,1,0)$, whose desingularization $\widetilde{S}(1,1,0)$ is isomorphic to $\mathbb{P}(O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}) \to \mathbb{P}^1 = L$. The surface $Y$ and $Z$ can be considered as Weil divisors of the normal 3-fold $S(1,1,0)$ whose pullbacks to $\widetilde{S}(1,1,0)$: $Y'$ and $Z'$ satisfy $Y' \sim 2H - R$ and $Z' \sim 2H + R$, respectively, where $H$ and $R$ denote the pullback of $O_{\mathbb{P}^4}(1)$ and the ruling over $L$, respectively.

Conversely, once we have a quadric equation $G$ of rank 4 in $\mathbb{I}_Y$, we obtain the situation : $X \subseteq Y \subseteq Q = \{G = 0\} \cong S(1,1,0)$. By choosing a Cartier divisor $Z'' \in |2H + R|$ suitably in the desingularization $\widetilde{S}(1,1,0)$ of $\{G = 0\}$, we obtain $X = Y'' \cap Z''$. Namely there exists an integral homological shell $\overline{Z}''$ of $X$ in $Q$ with $X = Y \cap \overline{Z}''$, which implies that $G \in \mathbb{I}_{\overline{Z}''}$, where the surface $\overline{Z}''$ is the image or the push down of the divisor $Z''$ in $\widetilde{S}(1,1,0)$.

Now let us go back to the situation of $\widehat{G}_Z$ and the morphism $\gamma(f)_1 = \gamma(f)_{(1,2)} : B \to \mathbb{P}^2_{(1,2)}$. As we mentioned above, the quadric equation $\widehat{G}_Z$ can also be written as $\widehat{G}_Z = \widehat{\tau}_0 G_0 + \widehat{\tau}_1 G_1 + \widehat{\tau}_2 G_2$. Then the morphism $\gamma(f)_{(1,2)} : B \to \mathbb{P}^2_{(1,2)}$ can be considered as:

$$\gamma(f)_{(1,2)} : B \ni Z \mapsto [\widehat{\tau}_0 : \widehat{\tau}_1 : \widehat{\tau}_2] \in \mathbb{P}^2_{(1,2)}.$$

From the consideration above, the image $Im(\gamma(f)_{(1,2)}) \subseteq \mathbb{P}^2_{(1,2)}$ of the morphism $\gamma(f)_{(1,2)}$ is the same as the set :

$$\{[\tau_0 : \tau_1 : \tau_2] \in \mathbb{P}^2_{(1,2)} \mid rank(G) = 4, G = \tau_0 G_0 + \tau_1 G_1 + \tau_2 G_2\}$$

By using the explicit equations (**??**), we can translate the quadric equation $G$ into a $5 \times 5$-matrix with the components written by $\{\tau_0, \tau_1, \tau_2\}$. The condition $rank(G) \leq 4$ is equivalent to $\tau_0 \cdot \tau_1 \cdot \tau_2 = 0$, and the condition $rank(G) \leq 3$ is the same as $\tau_0 = \tau_1 = 0$. Thus the image $D_0 = Im(\gamma(f)_{(1,2)}) \subseteq \mathbb{P}^2_{(1,2)}$ is expressed as :

$$D_0 = \{[\tau_0 : \tau_1 : \tau_2] \in \mathbb{P}^2_{(1,2)} \mid \quad \tau_0 \cdot \tau_1 \cdot \tau_2 = 0 \quad \} \setminus \{[0:0:1]\},$$

which is a quasiprojective reducible curve and is an open set of a union of 3 lines in $\mathbb{P}^2_{(1,2)}$ (cf. Figure **??**). Thus our parameter space $B$ is the fiber space over the curve $D_0$ by the morphism $\gamma(f)_{(1,2)}$.

Now we consider a fiber of the morphism $\gamma(f)_{(1,2)}$ precisely. Let us take a quadric equation $G \in D_0$ which defines a quadric hypersurface $Q = S(1,1,0) \subseteq \mathbb{P}^4 = P$ and two homological shells $Z_1$, $Z_2 \in \gamma(f)^{-1}_{(1,2)}(G) \subseteq B$. Namely, $Z'_i \subseteq \widetilde{Q} = \widetilde{S}(1,1,0)$ and $Z'_i \in |2H+R|$, where $\widetilde{Q} = \widetilde{S}(1,1,0)$ denotes the desingularization of the quadric hypersurface $Q = S(1,1,0)$ by blowing up at the vertex, or the only one singularity and $Z'_i$ is the pull back of $Z_i$ as a Weil divisor. Moreover we have $X = Z'_1 \cap Y' = Z'_2 \cap Y' \subseteq \widetilde{Q}$, where $Y'$ is also a pull back of $Y$ as a Weil divisor and $Y' \in |2H - R|$. Then we consider the natural short exact sequence:

$$0 \longrightarrow O_{\widetilde{Q}}(-2H+R) \xrightarrow[\sigma_{Y'}]{} O_{\widetilde{Q}} \longrightarrow O_{Y'} \longrightarrow 0,$$

where the section $\sigma_{Y'} \in H^0(O_{\widetilde{Q}}(2H-R))$ defines the divisor $Y'$. By tensoring $O_{\widetilde{Q}}(2H+R)$ to the sequence above, we obtain

$$0 \longrightarrow O_{\widetilde{Q}}(2R) \xrightarrow[\sigma_{Y'}]{} O_{\widetilde{Q}}(2H+R) \longrightarrow O_{Y'}(X) \longrightarrow 0,$$

which shows

$$H^0(O_{\widetilde{Q}}(2H+R)) \ni \sigma_{Z'_i} \mapsto \sigma_X \in H^0(O_{Y'}(X)),$$

where the sections $\sigma_{Z'_i}$ and $\sigma_X$ define the divisor $Z'_i$ on $\widetilde{Q}$ and the divisor $X$ on $Y'$, respectively. Now the difference of $\sigma_{Z'_1}$ and $\sigma_{Z'_2}$ comes from $H^0(O_{\widetilde{Q}}(2R)) \cong \mathbb{C}^3$, which means that the difference of the divisors $Z'_1$ and $Z'_2$ is parametrized by $\mathbb{P}^2$. Thus we have $\gamma(f)^{-1}_{(1,2)}(G) \cong \mathbb{P}^2$. In other words, the parameter space $B = B^w_4$ of the weak shell equivalence is the $\mathbb{P}^2$-fiber space over the curve $D_0$, which is connected. This shows that any two homological shells of the canonical curve $X$ with dimension 2 and degree 5 are weakly shell equivalent to each other.

Now we consider the parameter spaces in $B^s_4$ for the strict shell equivalences. Take any connected parameter space $B^s_{4,[Z]}$ in $B^s_4$ of a strict shell equivalence class which includes a homological shell $Z$ of the curve $X$ whose dimension and degree are 2 and 5, respectively. At least, the parameter space $B^s_{4,[Z]}$ is contained in a fiber of $\gamma$-map $\gamma(f) = \gamma(f)_{(1,2)} \times \gamma(f)_{(2,4)}$. A fiber $F \subseteq B$ of the map $\gamma(f)_{(1,2)}$ is isomorphic to $\mathbb{P}^2$, which shows that $B^s_{4,[Z]}$ is contained in the fiber of the morphism $\mu = \gamma(f)_{(2,4)}|_F : F \cong \mathbb{P}^2 \to \mathbb{P}^2_{(2,4)}$. Since the image of the morphism $\mu$ from $\mathbb{P}^2$ to $\mathbb{P}^2_{(2,4)}$ is the whole space $\mathbb{P}^2_{(2,4)}$ or the one point. Thus the morphism $\mu$ is finite or the constant map. In other words, the fiber of the morphism $\mu$ is finite set or $\mathbb{P}^2$ itself. By using the connectedness of $B^s_{4,[Z]}$, we see that $B^s_4 = \sqcup \bullet$ or $B^s_4 = \sqcup \mathbb{P}^2$. This is what we know about the space $B^s_4$ at this stage. ∎

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