Homological shells of a projective variety with Δ-genus 0

Takeshi Usa
Dept. of Math. Univ. of Hyogo *

Abstract

This is the formal publication of the results announced in rather a private style at the mourning memorial symposium for Prof. Ogoma in Kochi 2004. In [19], we presented several conjectures to study a homological shell $W \subseteq P = \mathbb{P}^N(\mathbb{C})$ of a given projective subvariety $X \subseteq P$, namely a closed subscheme W which includes the variety X and satisfies "Tor injectivity condition" with respect to X. Among the conjectures, our main concern is a conjecture on Δ-genus inequality. Based on the results of [3] and [4], we accomplish the classification of homological shells of a variety of minimal degree initiated by [21], [22] and [23], which gives a supporting evidence for this conjecture.

Keywords: homological shell, Δ-genus, variety of minimal degree

§0 Introduction.

Our theme is the "geometric structure" of a projective embedding of a given variety X, namely studying the existence of intermediate ambient varieties with some good properties for the embedded variety X. This is philosophically similar to studying intermediate extension fields of the given finite algebraic extension field. As we saw in [17] and [19], from the view point of infinitesimal obstruction theory, one can also find some indications of further analogy to the Galois theory.

Several conjectures on the geometric structure of projective embeddings are given in our paper [19]. Some of them are mentioned again in the next section §1 for adding some remarks.

One of the key concepts appeared in these conjectures is "homological shell"(cf. Definition 2.2), which is a special kind of intermediate ambient scheme for the given variety X and was first introduced in [18]. Homological shells inherit many excellent properties (cf. Proposition 2.6) from their homological core X (cf. Definition 2.2), which reflect the structure of higher syzygies of the homogeneous coordinate ring of the homological core X. We make a remark that homological shells had appeared implicitly in many classical works (cf. [12], [14], [6], [8], [15], [7],[2], [10], [13] etc.) as actual examples, which also suggests the importance of this concept.

In spite of many excellent properties of homological shells, there are still mysteries on homological shells (cf. [19]). For example, the conjectures in §1 are still open. In particular, we are interested in what we call "Δ-genus inequality conjecture" since this conjecture implies the same best possible upper bound of the degree of equations of the given variety as the one predicted by Eisenbud-Goto conjecture.

*2167 Shosha, Himeji, 671-2201 Japan.
E-mail address : usa@sci.u-hyogo.ac.jp
Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX

Fix a closed subvariety $X \subseteq \mathbb{P}^N(\mathbb{C})$ as a homological core, its homological shells generally exist infinitely many, but by the elementary properties of homological shells (cf. Proposition 2.6 (2.6.5)), they are bounded by an algebraic family of finite components, which suggests the theoretical possibility of classifying its homological shells completely. Thus, to find evidences for this conjecture, what we should do first is to classify all the homological shells of the variety X with assuming that its Δ-genus is zero and to show that any homological shell of the variety X is again a variety of Δ-genus zero.

This classification was started in [21], [22] and [23]. Since we classified by force all the intermediate ambient schemes with specified Hilbert polynomials by admitting non reduced structures or reducibility, except applying the Theory of Δ-genus (cf. [7]), almost of the techniques used in those articles are very primitive. However, we should make a remark here that those techniques can be also applied to classifying explicitly the homological shells of an actual low codimensional variety X which may not be 2-regular(cf. [24], [25]). Adding the aid of the marvelous results of [3] and [4] to our previous techniques, we can complete the classification on the homological shells of a variety of Δ-genus zero in Corollary 3.2.

The results presented here are announced at the mourning memorial symposium for Prof. Ogoma in Kochi 2004. The author has to apologize for the long delay of the formal publication of the results, which was caused by the change of the previous plan on the publication relating with the classification of homological shells of the canonical curves with small genera such as in [24] and [25].

§1 Conjectures.

Let us recall some of the conjectures given in the paper [19]. For precise definitions on our terminology, see the next section (cf. in particular, Definition 2.2).

Conjecture 1.1 *We fix an N-th projective space $P = \mathbb{P}^N(\mathbb{C})$ with the tautological ample line bundle $O_P(1) = O_P(H)$ and consider its closed subschemes.*

(1.1.1) *Assume that a closed subscheme $W \subseteq P$ is a homological shell of a closed subvariety $V \subseteq P$. Then the subscheme W is also a variety, namely reduced and irreducible.*

(1.1.2) **[Δ-genus inequality conjecture]** *Take closed subvarieties $V \subseteq W \subseteq P$. If W is a homological shell of V, then the inequality: $\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$ holds on their Δ-genera (e.g. $\Delta(V, O_V(1)) := dim(V) + deg(O_V(1)) - h^0(V, O_V(1))$;* cf. [7]).

Under the circumstances of Conjecture 1.1, we make some remarks on each claim above, respectively. We refer to Proposition 2.6 for preparatory knowledge on homological shells.

Remark 1.2 *We summarize what we know on the claim (1.1.1).*

(1.2.1) *This claim is obviously true if the variety V is a complete intersection.*

(1.2.2) *If the subscheme W is a scheme of codimension one, this claim is true for a general variety V.*

(1.2.3) *Theorem 3.1 gives another evidence of this claim.*

Remark 1.3 *Now we consider the claim (1.1.2).*

(1.3.1) *Let us consider the case that the scheme W is a divisor of P and is a homological shell of V. We may assume that the variety V is non-degenerate. Then, by Proposition 2.6, W is a member of minimal generators of $\mathbb{I}_V$. The Δ-genus inequality of (1.1.2) is the same as : $deg(W) + N - 1 - (N+1) \leq \Delta(V, O_V(1))$, namely $deg(W) \leq \Delta(V, O_V(1)) + 2$. This upper bound is the same as the one predicted by Eisenbud-Goto conjecture (cf. [5]). If we assume moreover that V is non-degenerate arithmetically Buchsbaum, then the result [16] shows that this upperbound is true.*

(1.3.2) *If the polarized manifold $(V, O_V(1))$ is arithmetically normal and is a hypersurface cut of the polarized manifold $(W, O_W(1))$, then W is a homological shell of V and this inequality is obviously true.*

(1.3.3) *As an interim report on giving evidences for Δ-genus inequality conjecture, one introduces a fact with relating to FG-shells(cf. Definition 2.2), whose proof will be presented in a forthcoming paper. Let $W \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a non-degenerate non-singular projective variety. Assume that $k := arith.depth(W) \geq 3$. For a positive integer q with $q \leq k-2$, we take integers $\{m_1, \ldots m_q, s\}$, with $m_1 \geq m_2 \geq \ldots m_q \geq 2$ and $q \geq s \geq 1$. Suppose that on the variety W, we have a regular section $\tau \in H^0(W, F)$ of the vector bundle $F = \oplus_{j=1}^{q} O_W(m_j)$ of rank q and a vector bundle E of rank s with an exact sequence:*

$$0 \longrightarrow G \longrightarrow F \xrightarrow{\varphi} E \longrightarrow 0.$$

We set $\sigma := \varphi(\tau)$ and the closed scheme V of W to be the zero locus $Z(\sigma)$ and assume that $arith.depth(V) \geq 2$. Then $(W, (E, \sigma))$ is a FG-shell of V and $\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$.

Here we make a remark on a conjecture reaised in [21] which relates closely to Corollary 3.2 and does not appear in Conjecture 1.1.

Remark 1.4 *In [21], we gave the following conjecture on closed schemes of an N-th projective space $P = \mathbb{P}^N(\mathbb{C})$, which is obviously true by Auslander-Buchsbaum formula if $dim(W) = dim(V)$.*

> *Assume that a closed subscheme $W \subseteq P$ is a homological shell of an arithmetically Cohen-Macaulay closed subscheme $V \subseteq P$. Then the subscheme W is also an arithmetically Cohen-Macaulay subscheme.*

Recently we have found an easy counter-example to this conjecture. For example, take integers $m_1 \geq m_2 \geq 2$ and a complete intersection variety Y of the type (m_1, m_2) defined by equations F_1 and F_2, where $deg(F_i) = m_i$ $(i = 1, 2)$. Then we set V to be the first infinitesimal neighborhood $(|Y|, O_P/I_Y^2)$ and W to be a closed scheme defined by the homogeneous ideal $\mathbb{I}_W := (F_1^2, F_1F_2) = (F_1) \cap (F_1, F_2)^2$. It is easy to check that W is a homological shell of V, V is arithmetically Cohen-Macaulay and W is not arithmetically Cohen-Macaulay.

§2 Preliminaries.

To avoid needless confusions, let us confirm our notation used in this article.

Notation and Conventions 2.1 *In this paper, we use the terminology of [9] without mentioned, and always admit the conventions and use the notation below for simplicity.*

(2.1.1) *Every object under consideration is defined over the field of complex numbers $\mathbb{C}$. We will work in the category of algebraic schemes and algebraically holomorphic morphisms (or rational maps) or in the categories of coherent sheaves and their ($\mathcal{O}$-linear) homomorphisms otherwise mentioning so.*

(2.1.2) *Let us take a complex projective scheme X of dimension n and one of its embeddings $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ $(N \geq 1)$. The sheaf of ideals defining $j(X)$ in P and the conormal sheaf are denoted by I_X and $N^\vee_{X/P} = I_X/I_X^2$, respectively. Taking a $\mathbb{C}$-basis $\{Z_0, \dots, Z_N\}$ of $H^0(P, O_P(1))$. Then we put:*

$$\begin{aligned} S &:= \bigoplus_{m \in \mathbb{Z}} H^0(P, O_P(m)) \cong \mathbb{C}[Z_0, \dots, Z_N] \\ S_+ &:= \bigoplus_{m > 0} H^0(P, O_P(m)) \cong (Z_0, \dots, Z_N)\mathbb{C}[Z_0, \dots, Z_N] \\ \widetilde{R_X} &:= \bigoplus_{m \in \mathbb{Z}} H^0(X, O_X(m)) \\ \mathbb{I}_X &:= \bigoplus_{m \in \mathbb{Z}} H^0(P, I_X(m)) \\ R_X &:= Im[S \to \widetilde{R_X}] \cong S/\mathbb{I}_X. \end{aligned}$$

In this case, the induced ample line bundle $j^*O_P(1) = j^*O_P(H)$ is denoted by $O_X(1)$ or $O_X(H)$.

Now let us recall our key concepts for studying the geometric structures of projective embeddings, the two of them, namely "H-shell" and "G-shell", were introduced first in [18]. On G-shells, we have slightly modified its definition from $V \subseteq Reg(W)$ in [18] to local Tor injectivity condition of (2.2.1) (cf. Remark 2.4 and [22]). By reconsidering classical examples in Complex Projective Geometry including *varieties of minimal degree*, standing on this new point of view, we can find many good actual examples for these concepts in a number of classical works such as [12], [14], [6], [8], [15], [7], [2], [10], [13] and so on.

Definition 2.2 (shells and cores) *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$ (namely the inclusion of the defining ideal sheaves: $I_V \supseteq I_W$ in the structure sheaf O_P of P; In this case, the subscheme W is called simply an* intermediate ambient scheme *of V).*

(2.2.1) *If the natural map:*

$$\mu_{q,x} : Tor_q^{O_{P,x}}(O_{W,x}, O_{P,x}/\mathfrak{m}_{P,x}) \to Tor_q^{O_{P,x}}(O_{V,x}, O_{P,x}/\mathfrak{m}_{P,x})$$

is injective for any integer $q \geq 0$ and for any point x on V (abbr. "local Tor injectivity condition"*), we say that W is a* local shell *of V and that V is a* local core *of W, where $\mathfrak{m}_{P,x}$ denotes the maximal ideal of the local ring $O_{P,x}$.*

(2.2.2) *If the natural map:*

$$\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ (abbr. "Tor injectivity condition"*), we say that W is a* homological shell *(abbr.* H-shell*) of V and that V is a* homological core *(abbr.* H-core*) of W.*

(2.2.3) *If V and W are closed subvarieties, W is a local shell, and a homological shell of V, then we say that W is a* geometric shell *(abbr.* G-shell*) of V and V is a* geometric core *(abbr.* G-core*) of W.*

(2.2.4) *Assume that (a) V is a closed subvariety and W is a geometric shell of V; (b) there exist a vector bundle E on W and a global section $\sigma \in \Gamma(W, E)$; (c) the zero locus $Z(\sigma)$ (the scheme structure of $Z(\sigma)$ is defined by an ideal sheaf: $I_{Z(\sigma)} := Im[\sigma : E^\vee \to O_P]$) coincides with V including their scheme structures; (d) by trivializing the bundle E locally, the section σ gives an O_W-regular sequence. Then we say that W (or more precisely $(W, (E, \sigma))$) is a* framed geometric shell *(abbr.* FG-shell*) of V and V is a* framed geometric core *(abbr.* FG-core*) of W. In this case, the pair (E, σ) is called a* shell frame *of V in W.*

For the subscheme V, the total space P and V itself are called trivial H-shells *(or* trivial G-shells *if V is a variety).*

Remark 2.3 *Homological shell defined in (2.2.2) above was called as* Pregeometric shell *or* PG-shell *in our previous works after we introduced this concept in [18].*

Remark 2.4 *On the definition (2.2.1) of local shell, let us consider the spacial cases such as V is smooth at the point x or V is locally complete intersection at the point x. Then, if W is a local shell of V, we see that W is smooth at the point x or W is a locally complete intersection at the point x, respectively. Thus the condition that W is a local shell of V implies that $Reg(V) \subseteq Reg(W)$.*

We should make a remark that there is a little difference on the view point between our research on the projective embeddings and existing theories on polarized varieties. Our view point is rather near to the classical Projective Geometry but has main concern on studying precisely intermediate ambient varieties by assuming arithmetic D_2-condition on the H-cores.

Remark 2.5 *With respect to a polarized variety (V, L), namely a pair of a projective variety V and an ample line bundle L on V, the main concern of Fujita's theory or of the theory of Δ-genus is in the geometric analysis of the embedding into a weighted projective space associated to the graded ring $G(V, L)$ instead of the embedding into a usual projective space by the linear system $|L|$. However, if we restrict ourselves to the case that the line bundle L is simply generated, or the variety V is embedded into a usual projective space with the "arithmetic D_2"condition, Fujita's theory can be applied directly to our problems. Thus, in this article, if we say that the subvariety $V \subset P = \mathbb{P}^N(\mathbb{C})$ is* a variety of Δ-genus zero *or* a (non-degenerate) variety of minimal degree*, it means that the pair $(V, O_V(1))$ is a variety of Δ-genus zero in the sense of Fujita's theory and the* *variety V is embedded into P by the comlete linear system $|O_V(1)|$ (and therefore the variety V is non-degenerate and linearly normal in P, which means $h^0(V, O_V(1)) = N+1$), where the simple generation of the (very) ample line bundle $O_V(1)$ is always guaranteed by Fujita's theory for the variety $(V, O_V(1))$ of Δ-genus zero (cf. [7]).*

Let us recall a proposition on several elementary properties of H-shells from [19], [21] and [22]. On the properties of G-shells related to their restricted syzygy bundles and infinitesimal syzygy bundles, which are not presented here, see [20].

Proposition 2.6 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$.*

(2.6.1) *If W is a hypersurface, then W is a homological shell of V if and only if the equation of W is a member of minimal generators of the homogeneous ideal $\mathbb{I}_V$ of V.*

(2.6.2) *Let V be a reduced and irreducible closed subscheme of P, a closed subscheme W is of codimension 1 in the total space P and a homological shell of the variety V. Then W is an reduced and irreducible divisor of P.*

(2.6.3) *Assume that the subscheme V is a complete intersection. Then the scheme W is a homological shell of V if and only if the subscheme W is defined by a part of minimal generators of $\mathbb{I}_V$.*

(2.6.4) *Take a closed scheme Y such that $V \subseteq Y \subseteq W$. Assume that W is a homological shell of V. Then W is also a homological shell of Y. In particular, the subscheme W is also a homological shell of the m-th infinitesimal neighborhood $Y = (V/W)_{(m)}$ of V in W, where $(V/W)_{(m)} = (|V|, O_W/I^{m+1}_{V/W})$.*

(2.6.5) *Fix the subscheme V of $codim(V, P) \geq 2$. Then all non-trivial homological shells of V form a non empty algebraic family of finite components (N.B. The family of all non-trivial G-shells of V may be empty even if V itself is a smooth variety).*

(2.6.6) *If W is a homological shell of V, then we have an inequality: $arith.depth(V) \leq arith.depth(W)$ on their arithmetic depths. In particular, if the natural restriction map $H^0(P, O_P(m)) \to H^0(V, O_V(m))$ is surjective for all integers m (i.e. $R_V = \widetilde{R_V}$), then the natural restriction map $H^0(P, O_P(m)) \to H^0(W, O_W(m))$ is also surjective for all integers m (i.e.$R_W = \widetilde{R_W}$). In other words, the arithmetic D_2 condition is inherited from homological cores to their homological shells.*

(2.6.7) *If $arith.depth(V) \geq 2$ and the subscheme W is a homological shell of the subscheme V, then we have an inequality on their Castelnuovo-Mumford regularity(cf. [5]): $reg^{CaM}(V) \geq reg^{CaM}(W)$.*

(2.6.8) *Assume that there exist r hypersurfaces $D_1, \ldots, D_r$ in P with homogeneous equations $F_1, \cdots, F_r$ of degree $m_1, \cdots, m_r$, respectively, and satisfying the conditions : (a) $V = W \cap D_1 \cap \ldots \cap D_r$; (b) $H^0(W_t, O_{W_t}) = \mathbb{C}$ $(t = 0, \cdots, r)$, where $W_0 := W$ and $W_t := W \cap D_1 \cap \ldots \cap D_t$ $(t = 1, \cdots, r)$; (c) the homogeneous equations $F_1, \cdots F_r$ form an O_W-regular sequence, namely the sequence :*

$$0 \longrightarrow O_{W_{t-1}}(-m_t) \xrightarrow{\times F_t} O_{W_{t-1}},$$

is exact for $t = 1, \cdots, r$. If $arith.depth(V) \geq 2$, then W is a homological shell of V.

(2.6.9) *Assume that the subscheme V is non-degenerate, namely no hyperplane contains V. If W is a 2-regular scheme (more precisely, its homogeneous ideal $\mathbb{I}_W$ is 2-regular e.g. a variety of minimal degree), namely the homogeneous coordinate ring R_W of W has a minimal S-free resolution of the form : $0 \leftarrow R_W \leftarrow S \leftarrow F_1(-2) \leftarrow F_2(-3) \leftarrow \cdots \leftarrow F_p(-p-1) \leftarrow \cdots$ (cf. [5]), where $F_u(v)$ denotes $\oplus S(v)$: a direct sum of several copies of S with degree v shift, then W is a homological shell of V.*

(2.6.10) *Assume that there is a hyperplain $H \subset P$ including both the schemes V and W. Then the scheme W is a homological shell of V in H if and only if the scheme W is a homological shell of V in P.*

(2.6.11) *Assume that arith.depth$(V) \geq 2$ and the scheme W is a homological shell of V in P. If we take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element, then the scheme $W \cap H$ is a homological shell of $V \cap H$ in the projective space $H \cong \mathbb{P}^{N-1}(\mathbb{C})$ (or in P).*

(2.6.12) *Suppose that $H^0(O_V) = H^0(O_W) = \mathbb{C}$. Take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element. Assume that arith.depth$(V \cap H) \geq 2$ and the scheme $W \cap H$ is a homological shell of $V \cap H$ in the projective space H (or in P). Then the scheme W is a homological shell of V in P.*

Remark 2.7 *To handle homological shells of codimension one, before applying the claim (2.6.1), we should take care of the fact that for an arbitrary closed scheme $V \subsetneq P$, the condition that codim$(W) = 1$ and the scheme W is a homological shell of the scheme V does not in general imply that the scheme W is a divisor of P since we do not assume, for example, the scheme W is equidimensional and so on. It may happen that the scheme W has a primary component of codimension 1 and has another component of codimension more than 1 or an embedded component (e.g. see Remark 1.4). Thus, to consider homological shells of codimension 1 for a variety V, we need the claim (2.6.2).*

Related to Proposition 2.6 above, we have to make a remark that there were two minor mistakes of the failure to attach some assumptions in the claims of [19]. In case of considering arithmetically normal subvarieties, which is our main concern, there is no effect on our consideration. However, when we apply hyperplain cut method to the classification of H-shells including schemes as in [22] or in [23], we need careful consideration on these assumptions.

Remark 2.8 *The first mistake was in the old version of the claim (2.6.6) (cf. [19]). The author had failed to attach the condition: arith.depth$(V) \geq 2$ by a misprint, which is corrected in the improved version of [19]:math.AG/0001004. The second one was a failure to attach the condition: $H^0(W_t, O_{W_t}) = \mathbb{C}$ $(t = 0, \cdots, r)$ of the claim (2.6.8). Without assuming this condition, the claim (2.6.8) is not true in general. A counterexample is given in [22].*

For the variety of minimal degree, relating to Conjecture 1.1 (1.1.1), we have a very easy but also useful result, which simplifies our argument later.

Proposition 2.9 *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a variety of Δ-genus zero (cf. Remark 2.5) whose dimension is n, and a scheme $W \subseteq P$ a homological shell of V. Assume that the scheme W is arithmetically Cohen-Macaulay and of (pure) dimension m. Then, the scheme W is also a variety of Δ-genus zero.*

Proof. By the results of [7] and [6], our assumption implies that the homogeneous coordinate ring R_V is Cohen-Macaulay and has a 2-linear minimal S-free resolution. The minimal S-free resolution of R_V is of the form: $\mathbb{F}_{V,\bullet} : 0 \leftarrow R_V \leftarrow \mathbb{F}_{V,0} = S \leftarrow \mathbb{F}_{V,1} = \oplus^{v(1)}S(-2) \leftarrow \mathbb{F}_{V,2} = \oplus^{v(2)}S(-3) \leftarrow \cdots \leftarrow \mathbb{F}_{V,p} = \oplus^{v(p)}S(-p-1) \leftarrow \cdots \leftarrow \mathbb{F}_{V,r} = \oplus^{v(r)}S(-r-1) \leftarrow 0$, where $r = N - n$. Since we assume that the scheme W is arithmetically Cohen-Macaulay and is a H-shell of V, the homogeneous coordinate ring R_W has a minimal S-free resolution of the form: $\mathbb{F}_{W,\bullet} : 0 \leftarrow R_W \leftarrow \mathbb{F}_{W,0} = S \leftarrow \mathbb{F}_{W,1} = \oplus^{w(1)}S(-2) \leftarrow \mathbb{F}_{W,2} = \oplus^{w(2)}S(-3) \leftarrow \cdots \leftarrow \mathbb{F}_{W,t} = \oplus^{w(t)}S(-t-1) \leftarrow 0$, where $t = N - m = r - (m - n)$. Then we apply Theorem 4.1.15 of [1] and see that $e := deg(V) = (1/r!)\Pi_{k=1}^{r}(k+1) = r+1$ and $e' := deg(W) = (1/t!)\Pi_{k=1}^{t}(k+1) = t+1$. Let us show that the scheme W is a variety. Take a main primary component : W_0 of W which includes V, give a reduced structure on W_0, and put $\overline{e_0} := deg((W_0)_{red})$. Since the scheme W is arithmetically Cohen-Macaulay, the scheme W is locally Cohen-Macaulay and equidimensional. And therefore, the scheme W is a variety if and only if $\overline{e_0} = e'$ (cf. [11]). Now we assume $\overline{e_0} < e'$. Then $\overline{e_0} < t+1 = N - m + 1 = N - dim((W_0)_{red}) + 1$, which implies that the variety $(W_0)_{red}$ is degenerate, namely there is a hyperplain H including the variety $(W_0)_{red}$. Then $V \subset (W_0)_{red}$ implies that the variety V is also degenerate, which contradicts our assumption (cf. Remark 2.5). ∎

Corollary 2.10 *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a variety of Δ-genus zero and a scheme $W \subseteq P$ a homological shell of V. Assume that* $\dim(V) = \dim(W)$. *Then,* $V = W$.

Proof. By the assumption and (2.6.5), $\dim(R_W) = \dim(R_V) = depth_{S_+}(R_V) \leq depth_{S_+}(R_W)$, which shows the ring R_W is Cohen-Macauilay. Then, applying Proposition 2.9, we see that the scheme W is a variety, which implies $V = W$. ∎

§3 A Result.

In April 2004, D. Eisenbud, M. Green, K.Hulek, and S. Popescu released their marvelous results on 2-regular schemes in two preprints [3] and [4]. Applying their results, we can easily obtain a general classification of homological shells of a variety of minimal degree.

Theorem 3.1 *Let $Y \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a non-degenerate closed subvariety, and $W \subseteq P$ a 2-regular closed subscheme including the variety Y. Then the scheme W is reduced and irreducible, and therefore a variety of minimal degree.*

Proof. Let us consider the triplet of closed schemes $Y \subseteq V \subseteq W$ in P, where $V := W_{red}$ is the reduced scheme of W. By Corollary 0.7 of [4], we see that $V = W_{red}$ is also a 2-regular scheme. Take the irreducible decomposition $V = V_1 \cup V_2 \cup \ldots V_m$ of V. Then, by Theorem 0.3 and Proposition 2.4 of [4], any irreducible component V_i of the scheme V is a variety of minimal degree in its linear span $span(V_i)$ and any two irreducible components V_i and V_j are linearly joined, namely $V_i \cap V_j = span(V_i) \cap span(V_j)$. Now we may assume that $Y \subseteq V_1$. By the reason that Y is non-degenerate, i.e. $P = span(Y) = span(V_1)$, we have $V_1 \supseteq V_1 \cap V_j = span(V_1) \cap span(V_j) = span(V_j) \supseteq V_j$ for $j = 2, 3 \ldots m$, namely $V_1 \supseteq V_j$ for $j = 2, 3 \ldots m$. In other words, the scheme $V = W_{red}$ is a variety of minimal degree in P. Since the scheme W is 2-regular and includes the variety V, the scheme W is a homological shell of V and $dim(W) = dim(V)$. Now we can apply Corollary 2.10, we obtain $W = V$. ∎

Corollary 3.2 *Let $X \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a variety of Δ-genus zero, namely a (non-degenerate) variety of minimal degree and $W \subseteq P$ a closed subscheme including the variety X. Then the scheme W is a homological shell of X if and only if the scheme W is also a variety of minimal degree.*

Proof. Since the variety X is 2-regular, the scheme W is a homological shell of X if and only if the scheme W is also 2-regular. ∎

§4 Comparison Problems.

Let us give a definition first to simplify our discussion on chains of homological shells and of homological cores.

Definition 4.1 *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a closed subscheme of dimension n. For increasing integers $1 \leq e_1 < e_2 < \ldots < e_p \leq n$ and $n < f_1 < f_2 < \ldots < f_q \leq N$, if there exists a chain of closed subschemes:*

$$Z_1 \subset Z_2 \subset \cdots \subset Z_p \subseteq V \subset W_1 \subset W_2 \subset \cdots\cdots \subset W_q \subseteq P,$$

where the scheme Z_s $(s = 1, \cdots, p)$ is a homological core of V with $dim(Z_s) = e_s$ and the scheme W_t $(t = 1, \cdots, q)$ is a homological shell of V with $dim(W_t) = f_t$, then we say that this chain is a homological chain (abbr. H-chain) *of type $(e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_q)$ for V. Without assuming the existence of the lower part of the chain, the upper part of the chain: $\{W_t\}_{t=1}^q$ is called* a homological shell chain (abbr. H-shell chain) *of type $(f_1, f_2, \ldots, f_q)$ for V. Similarly, without assuming the existence of the upper part of the chain, the lower part of the chain: $\{Z_s\}_{s=1}^p$ is called* a homological core chain (abbr. H-core chain) *of type $(e_1, e_2, \ldots, e_p)$ for V. If p=n, (resp. q=N-n, or p=N-q=n), then we say that the homological core chain $\{Z_s\}_{s=1}^n$ (resp. the homological shell chain $\{W_t\}_{t=1}^{N-n}$ or the homological chain $\{Z_s, W_t| \quad s = 1, \ldots, n; t = 1, \ldots, N-n\}$) is* full. *If all the members of a chain are varieties, then we say that the chain is* integral.

Now we recall the next result from [21] with adding the result of Corollary 3.2.

Proposition 4.2 *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a closed subvariety of dimension n and of Δ-genus zero. Then, there is a full integral homological chain for V :*

$$Z_1 \subset Z_2 \subset \cdots \subset Z_n = V \subset W_1 \subset W_2 \subset \cdots \subset W_t \subset \cdots \subset W_{N-r} = P,$$

where each variety W_t is of Δ-genus zero. Moreover, if we take any other homological shell chain for V:

$$V \subset \widetilde{W_1} \subset \widetilde{W_2} \subset \cdots \subset \widetilde{W_t} \subset \cdots \subset \widetilde{W_q} \subseteq P,$$

then each scheme $\widetilde{W_t}$ is also a variety of Δ-genus zero. Thus any homological shell chain of V is integral.

Remark 4.3 *Under the circumstances of the Proposition 4.2, we can not say that any homological core chain of V is integral (cf. Proposition 2.6 (2.6.4)).*

As we gave the Problem 3.7 in [22], we do not know whether or not for a given closed subvariety $V \subseteq P = \mathbb{P}^N(\mathbb{C})$, there always exists a homological full chain of V (cf. also Proposition 1.9 of [19]). On the other hand, even if we restrict ourselves to a variety of a special type, e.g. a variety of minimal degree, comparing two homological shell chains of the given closed subvariety $V \subseteq P$ is a harder problem than the existence problem. Thus we give a problem on comparing two homological shells as the starting point of comparison problems on two homological shell chains.

Problem 4.4 *Let $V \subsetneq P = \mathbb{P}^N(\mathbb{C})$ be a closed subvariety of dimension n. Assume that we have two varieties: W and $\widetilde{W}$ of minimal degree which include the variety V and $m := dim(W) < \widetilde{m} := dim(\widetilde{W})$. Then, does there exist a variety U of minimal degree which satisfies $dim(U) = m$ and $V \subset U \subset \widetilde{W}$? If this claim is not true, what about the conclusion when we assume moreover that there is a quotient sheaf E of the conormal sheaf $N^{\vee}_{V/\widetilde{W}}$ whose Hilbert polynomial $A_E(m)$ is the same as the Hilbert polynomial $A_{N^{\vee}_{V/W}}(m)$ of the conormal sheaf $N^{\vee}_{V/W}$, or more strongly assume that the sheaf E is isomorphic to the conormal sheaf $N^{\vee}_{V/W}$?*

We also give an example of comparison problems on two homological shell chains for our discussion in future.

Problem 4.5 *Let $V \subsetneq P = \mathbb{P}^N(\mathbb{C})$ be a closed variety of dimension n. For increasing integers: $n < f_1 < f_2 < \ldots < f_q \leq N$, let us suppose that we have two integral homological shell chains of type $(f_1, f_2, \ldots, f_q)$ for V:*

$$V \subset W_1 \subset W_2 \subset \cdots\cdots \subset W_q \subseteq P,$$

and

$$V \subset \widetilde{W_1} \subset \widetilde{W_2} \subset \cdots\cdots \subset \widetilde{W_q} \subseteq P.$$

Moreover, assume that for each $t = 1, \ldots, q$, two Hilbert polynomials: $A_{W_t}(m)$ and $A_{\widetilde{W_t}}(m)$ coincide with each other and there is a sheaf isomorphism : $N^{\vee}_{V/W_t} \cong N^{\vee}_{V/\widetilde{W_t}}$. Then does there exist "a flat family of homological shell chains" *connecting the given two chains: $\{W_t\}_{t=1}^q$ and $\{\widetilde{W_t}\}_{t=1}^q$, namely a chain of closed subschemes $\{\mathcal{W}_t\}_{t=1}^q$ which are flat over a connected algebraic scheme T:*

$$V \times T \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots\cdots \subset \mathcal{W}_q \subseteq P \times T$$

such that for any closed point $x \in T$, each fibre $\mathcal{W}_t(x)$ $(t = 1, \ldots, q)$ over the point x is a homological shell of dimension f_t for V, and there exist two closed points $x_0, y_0 \in T$ satisfying $\mathcal{W}_t(x_0) = W_t$ and $\mathcal{W}_t(y_0) = \widetilde{W_t}$ $(t = 1, \ldots, q)$?

References

[1] W. Bruns and J. Herzog : Cohen-Macaulay Rings, Cambridge studies in advanced mathematics 39, Cambridge University Press, (1993).

[2] D. Eisenbud : Green's conjecture, in "Free Resolutions in Commutative Algebra and Algebraic Geometry", (Editors) D. Eisenbud et al. Boston, MA (1992).

[3] D. Eisenbud, M. Green, K.Hulek, and S. Popescu : Restricting linear suzygies: algebra and geometry, Duke E-print math.AG/0404516.

[4] D. Eisenbud, M. Green, K.Hulek, and S. Popescu : The geometry of 2-regular algebraic sets, Duke E-print math.AG/0404517.

[5] D. Eisenbud and S. Goto : Linear free resolutions and minimal multiplicity, J. of Algebra 88, pp. 89-133 (1984).

[6] D. Eisenbud, O. Riemenschneider and F. O. Schreyer : Resolutions of Cohen-Macaulay algebras, Math.Ann.257, pp. 85-98 (1981).

[7] T. Fujita : Classification theories of polarized varieties, London Math. Soc. Lect. Note Ser. 155, Cambridge Univ. Press (1990).

[8] M. Green : Koszul cohomology and geometry of projective varieties, J. Differ. Geom. 19, pp. 125-171 (1984).

[9] R. Hartshorne : Algebraic Geometry, GTM52, Springer-Verlag, (1977).

[10] R. Hartshorne: Varieties of small codimension in projective space, Bull. A. M. S., 80, pp. 1017-1032 (1974).

[11] S. L. Kleiman : Toward a numerical theory of ampleness, Ann. of Math., 84 pp. 293-344 (1966).

[12] K. Petri : Über die invariante Darstellung algebraischer Funktionen einer Variablen, Math. Ann. 88, pp. 243-289 (1923).

[13] Z. Ran : Absence of the Veronese from Smooth Threefolds in $\mathbb{P}^5$, Contemp. Math. 116, pp.125-128 (1991).

[14] B. Saint-Donat : On Petri's analysis of the linear system of quadrics through a canonical curve, Math. Ann. 206 pp. 157-175 (1973).

[15] F. O. Schreyer : Syzygies of canonical curves and special linear series, Math. Ann. 275, pp. 105-137 (1986).

[16] J.Stückrad and W. Vogel : Castelnuovo bounds for locally Cohen-Macaulay schemes. Math. Nachr. 136, pp. 307-320 (1988).

[17] T. Usa : Obstructions of infinitesimal lifting, Comm. Algebra, 17(10), pp. 2469-2519 (1989).

[18] T. Usa : An example of filtrations on syzygies induced by meta-Lefschetz operators, Report Sci. H. I. T., No.8, pp. 12-25 (1997).

[19] T. Usa : Problems on geometric structures of projective embeddings, Report Sci. H. I. T. , No.9, pp. 12-29 (1998) (Improved version : arXive e-print math.AG/0001004).

[20] T. Usa : Infinitesimal Syzygy Bundles, Report Sci. H. I. T. , No.12, pp. 1-17 (2001).

[21] T. Usa : A Classification of Pregeometric Shells of a Rational Quartic Curve, Report Sci. H. I. T. , No.13, pp. 1-17 (2002).

[22] T. Usa : Pregeometric Shells of a Veronese surface, Report Sci. H. I. T., No.14, pp. 1-15 (2003).

[23] T. Usa : Pregeometric Shells of a Rational Quintic Curve, Report Univ. of Hyogo, No.15, pp.1-16 (2004).

[24] T. Usa : Pregeometric Shells of a Canonical Curve of genus ≤ 5, Report of Univ. of Hyogo, No.20, pp. 1-15 (2009).

[25] T. Usa : Homological shell surfaces with degree 5 of a trigonal canonical curve of genus 5, Report of Univ. of Hyogo, No.21, pp. 1-21 (2010).

(Received October 28, 2011 ; Accepted December 12, 2011)