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Pregeometric Shells of a Canonical Curve of genus ≤ 5

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Abstract

To get more evidences for general conjectures raised in [11], which are on Δ-genus inequality and on the integrality of pregemetric shells (cf. Conjecture 1.1), we investigate the pregeometric shells of a canonical curve of genus ≤ 5.
Keywords: pregeometric shell, canonical curve, trigonal curve, genus ≤ 5

§0 Introduction.

Our main concern is to understand the "geometric structure" of a projective embedding of a given variety X, namely to study intermediate ambient schemes satisfying the "global Tor injectivity condition", i.e. a certain good condition from the view point of syzygies for the embedded variety X. Such an intermediate ambient scheme is called "a pregeometric shell" (abbr. PG-shell), whose precise definition is given in Definition 3.1, and was first introduced in [10]. Related with this concept, we raised several problems and conjectures in [11], which include the conjecture on an inequality for Δ-genera and the conjecture on integrality of pregemetric shells (cf. Conjecture 1.1).

To obtain evidences for these two conjectures, we started a project of classifying pregeometric shells in many classical examples of projective embeddings where the minimal free resolutions of the homogeneous coordinate rings are already known. For example, in [13], [14], [15] and [16], we classified all the pregeometric shells of given embedded varieties X of Δ-genus zero (of minimal degree). In all these cases, we obtained affirmative results for the both conjectures.

Thus, in this article, we carry out the classification for pregeometric shells of canonical curves of genera ≤ 5, explicitly. In these cases, the minimal free resolutions of the homogeneous coordinate rings of the canonical curves are not in general of pure degree and not of Koszul type, which attracts our paticular interest. Among them, the most interesting cases are the cases of trigonal cases of genus 5 canonical curves, which need more refined analysis on the minimal free resolutions than those already obtained in classical references such as [8] and [2]. After all, we can confirm that our two conjectures mentioned above hold also for canonical curves of genera ≤ 5 (cf. Main Theorem 2.1) and can find a new type of pregeometric shell (cf. Remark 4.1).

Also in this article, we use successively the notation and conventions in [13] without mention.

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§1 Conjectures.

Let us recall two conjectures on pregeometric shells from [11] and [13]. See Definition 3.1 for precise definitions on our terminology.

Conjecture 1.1 *Let $P = \mathbb{P}^N(\mathbb{C})$ be an N-th projective space with the tautological ample line bundle $O_P(1) = O_P(H)$ and $V \subseteq W \subseteq P$ its closed subschemes.*

(1.1.1) *Assume that the scheme V is a variety, namely reduced and irreducible and that the closed subscheme W is a pregeometric shell of V. Then the subscheme W is also a variety.*

(1.1.2) [**Δ-genus inequality conjecture**] *Suppose that the both subschemes V and W are varieties. If W is a pregeometric shell of V, then the inequality: $\Delta(V, O_V(1)) \geq \Delta(W, O_W(1))$ holds on their Δ-genera (cf. $\Delta(V, O_V(1)) := dim(V) + deg(O_V(1)) - h^0(V, O_V(1))$;* see [4]*).*

Remark 1.2 *On the conjecture (1.1.2), we can define a Δ-genus for a pair of a scheme and an ample invertible sheaf by using its Hilbert polynomial to define the degree, and formulate this conjecture without assuming integrality of V and of W. For additional information on these two conjectures, see §1 of [15].*

§2 Main Results.

Let us give an overview by summarizing our main results in this article as a main theorem. Here we should emphasize again that our "schemes" of course may have a non-equidimensional component or a non-reduced structure.

Main Theorem 2.1 *Let $X \subseteq P = \mathbb{P}^{g-1}(\mathbb{C})$ be a canonical curve of genus $g \leq 5$. Namely, taking a non hyperelliptic curve C of genus $3 \leq g = g(C) \leq 5$ and its canonical embedding $\Phi_{|K_C|} : C \to P = \mathbb{P}^{g-1}(\mathbb{C})$, we set $X := \Phi_{|K_C|}(C)$. Suppose that a closed subscheme $W \subseteq P$ is a pregeometric shell of X. Then the scheme W is always arithmetically Cohen-Macaulay and is a variety, namely irreducible and reduced. Moreover, the inequality : $\Delta(W, O_W(1)) \leq \Delta(X, O_X(1)) = \Delta(C, K_C) = g - 1$ holds, where the ample line bundles $O_W(1)$ and $O_X(1)$ are the restrictions of the ample tautological line bundle $O_P(1) = O_P(H)$.*

§3 Preliminaries.

In this section, we prepare several facts and concepts which appeared already in our previous papers : [11] $\sim$ [16].

First we recall our key concepts for studying the geometric structures of projective embeddings, the two of them, namely "PG-shell" and "G-shell", were introduced first in [10]. By reconsidering classical examples in Complex Projective Geometry including *varieties of minimal degree*, standing on this new point of view, we can find many good actual examples of these concepts in a number of classical works such as [6], [7], [8], [4] and so on.

Definition 3.1 (shells and cores) *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$ (namely the inclusion of the defining ideal sheaves: $I_V \supseteq I_W$ in the structure sheaf O_P of P; In this case, the subscheme W is called simply an* intermediate ambient scheme *of V).*

(3.1.1) *If the natural map:*

$$\mu_{q,x} : Tor_q^{O_{P,x}}(O_{W,x}, O_{P,x}/\mathfrak{m}_{P,x}) \to Tor_q^{O_{P,x}}(O_{V,x}, O_{P,x}/\mathfrak{m}_{P,x})$$

is injective for any integer $q \geq 0$ and for any point x on V (abbr. "local Tor injectivity condition"*), we say that W is a* local shell *of V and that V is a* local core *of W, where $\mathfrak{m}_{P,x}$ denotes the maximal ideal of the local ring $O_{P,x}$.*

(3.1.2) *If the natural map:*

$$\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_V, S/S_+)$$

is injective for every integer $q \geq 0$ (abbr. "global Tor injectivity condition"*), we say that W is a* pregeometric shell *(abbr.* PG-shell*) of V and that V is a* pregeometric core *(abbr.* PG-core*) of W.*

(3.1.3) *If the schemes V and W are closed subvarieties, the variety W is a local shell of V, and also a pregeometric shell of V, then we say that the variety W is a* geometric shell *(abbr.* G-shell*) of V and the variety V is a* geometric core *(abbr.* G-core*) of W.*

For the subscheme V, the total space P and V itself are called trivial PG-shells *(or* trivial G-shells *if V is a variety).*

Remark 3.2 *On geometric shells, we have slightly modified in [14] its definition from the one including the condition: $V \subseteq Reg(W)$ in [10] to the other one including the condition of local Tor injectivity as in (3.1.3). On the definition (3.1.1) of local shell, let us consider the spacial cases such as V is smooth at the point x or V is locally complete intersection at the point x. Then, if W is a local shell of V, we see that W is smooth at the point x or W is a locally complete intersection at the point x, respectively. Thus the condition that the scheme W is a local shell of V implies that $Reg(V) \subseteq Reg(W)$.*

Now, in the following proposition, we review several elementary properties of PG-shells (for their proves, see [11]). On the properties of G-shells related to their restricted syzygy bundles and infinitesimal syzygy bundles, which are not presented here, see [12].

Proposition 3.3 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$.*

(3.3.1) *If W is a hypersurface of P, then W is a pregeometric shell of V if and only if the equation of W is a member of minimal generators of the homogeneous ideal $\mathbb{I}_V$ of V.*

(3.3.2) *Let V be a reduced and irreducible closed subscheme of P, a closed subscheme W is of codimension 1 in the total space P and a pregeometric shell of the variety V. Then W is of pure codimension* 1, *namely a divisor of P, reduced and irreducible .*

(3.3.3) *Assume that the subscheme V is a complete intersection. Then the scheme W is a pregeometric shell of V if and only if the subscheme W is defined by a part of minimal generators of $\mathbb{I}_V$.*

(3.3.4) *Take a closed scheme Y such that $V \subseteq Y \subseteq W$. Assume that W is a pregeometric shell of V. Then W is also a pregeometric shell of Y. In particular, the subscheme W is also a pregeometric shell of the m-th infinitesimal neighborhood $Y = (V/W)_{(m)}$ of V in W, where $(V/W)_{(m)} = (|V|, O_W/I_{V/W}^{m+1})$.*

(3.3.5) *Fix the subscheme V of $codim(V,P) \geq 2$. Then all non-trivial pregeometric shells of V form a non empty algebraic family of finite components (N.B. The family of all non-trivial G-shells of V may be empty even if V itself is a smooth variety).*

(3.3.6) *If W is a pregeometric shell of V, then we have an inequality: $arith.depth(V) \leq arith.depth(W)$ on their arithmetic depths. In particular, if the natural restriction map $H^0(P, O_P(m)) \rightarrow H^0(V, O_V(m))$ is surjective for all integers m (i.e. $R_V = \widetilde{R_V}$), then the natural restriction map $H^0(P, O_P(m)) \rightarrow H^0(W, O_W(m))$ is also surjective for all integers m (i.e.$R_W = \widetilde{R_W}$). In other words, the arithmetic D_2 condition is inherited from pregeometric cores to their pregeometric shells.*

(3.3.7) *If $arith.depth(V) \geq 2$ and the subscheme W is a pregeometric shell of the subscheme V, then we have an inequality on their Castelnuovo-Mumford regularity(cf. [3]): $reg^{CaM}(V) \geq reg^{CaM}(W)$.*

(3.3.8) *Assume that there exist r hypersurfaces $D_1, \ldots, D_r$ in P with homogeneous equations $F_1, \cdots, F_r$ of degree $m_1, \cdots, m_r$, respectively, and satisfying the conditions : (a) $V = W \cap D_1 \cap \ldots \cap D_r$; (b) $H^0(W_t, O_{W_t}) = \mathbb{C}$ ($t = 0, \cdots, r$), where $W_0 := W$ and $W_t := W \cap D_1 \cap \ldots \cap D_t$ ($t = 1, \cdots, r$) ; (c) the homogeneous equations $F_1, \cdots, F_r$ form an O_W-regular sequence, namely the sequence :*

$$0 \longrightarrow O_{W_{t-1}}(-m_t) \xrightarrow{\times F_t} O_{W_{t-1}},$$

is exact for $t = 1, \cdots, r$. If $arith.depth(V) \geq 2$, then W is a pregeometric shell of V.

(3.3.9) *Assume that the subscheme V is non-degenerate, namely no hyperplane contains V. If W is a 2-regular scheme (more precisely, its homogeneous ideal $\mathbb{I}_W$ is 2-regular e.g. a variety of minimal degree), namely the homogeneous coordinate ring R_W of W has a minimal S-free resolution of the form : $0 \leftarrow R_W \leftarrow S \leftarrow F_1(-2) \leftarrow F_2(-3) \leftarrow \cdots \leftarrow F_p(-p-1) \leftarrow \cdots$ (cf. [3]), where $F_u(v)$ denotes $\oplus S(v)$: a direct sum of several copies of S with degree v shift, then W is a pregeometric shell of V.*

(3.3.10) *Assume that there is a hyperplain $H \subset P$ including both the schemes V and W. Then the scheme W is a pregeometric shell of V in H if and only if the scheme W is a pregeometric shell of V in P.*

(3.3.11) *Assume that $arith.depth(V) \geq 2$ and the scheme W is a pregeometric shell of V in P. If we take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element, then the scheme $W \cap H$ is a pregeometric shell of $V \cap H$ in the projective space $H \cong \mathbb{P}^{N-1}(\mathbb{C})$ (or in P).*

(3.3.12) *Suppose that $H^0(O_V) = H^0(O_W) = \mathbb{C}$. Take a hyperplain $H \subset P$ with a linear equation F which is an O_V and O_W-regular element. Assume that $arith.depth(V \cap H) \geq 2$ and the scheme $W \cap H$ is a pregeometric shell of $V \cap H$ in the projective space H (or in P). Then the scheme W is a pregeometric shell of V in P.*

Remark 3.4 *To handle pregeometric shells of codimension one, before applying the claim (3.3.1), we should take care of the fact that for an arbitrary closed scheme $V \subseteq P$, the condition that $codim(W) = 1$ and the scheme W is a pregeometric shell of the scheme V does not in general imply that the scheme W is a divisor of P since we do not assume, for example, the scheme W is equidimensional and so on. It may happen that the scheme W has a primary component of codimension 1 and has another component of codimension more than 1 or an embedded component (e.g. see an example in Remark 1.4 of [16]). Thus, to consider pregeometric shells of codimension 1 for a variety V, we need the claim (3.3.2).*

To handle Hilbert polynomials efficiently, we recall the following lemma from [13].

Lemma 3.5 (Finite factorial series expansion of Taylor type) *Let us consider a polynomial of real coefficients $f(x) \in \mathbb{R}[x]$ of degree r, in other words, a real valued function $f(x)$ defined on the field of real numbers $\mathbb{R}$ (or on the ring of rational integers $\mathbb{Z}$) which has an expression by factorial monomials $x^{[k]}$ $(k = 0, 1, \dots, r)$:*

$$\begin{aligned} f(x) &= p_0 + \left(\frac{p_1}{1!}\right) x^{[1]} + \left(\frac{p_2}{2!}\right) x^{[2]} + \cdots + \left(\frac{p_{r-1}}{(r-1)!}\right) x^{[r-1]} + \left(\frac{p_r}{r!}\right) x^{[r]} \\ &= p_r A_r(x) + p_{r-1} A_{r-1}(x) + \cdots + p_1 A_1(x) + p_0, \end{aligned}$$

where the coefficient p_k is also a real number, and the k-th factorial monomial $x^{[k]}$ means $(x+k)(x+k-1)(x+k-2)\cdots(x+1)$ and the Hilbert function $A_k(x)$ of $\mathbb{P}^k(\mathbb{C})$ has the form:

$$A_k(x) = \frac{x^{[k]}}{k!} = \begin{pmatrix} x+k \\ k \end{pmatrix} = \frac{(x+k)(x+k-1)\cdots(x+1)}{k!}.$$

Then the coefficient p_k can be computed by using the (backward) difference operator ∇ as follows.

$$p_k = (\nabla^k f)(-1)$$

As in [13], for a coherent O_P-module F on $P = \mathbb{P}^N(\mathbb{C})$, its Hilbert polynomial $\chi(F(m)) = A_F(m)$ is expressed in the form:

$$A_F(m) = \sum_{k=0}^{N} p_k(F) A_k(m),$$

where $p_k(F)$ denotes a suitable coefficient determined by the procedure above. For a closed subscheme $V \subseteq P$, we use the symbols: $A_V(m)$ and $p_k(V)$ instead of $A_{O_V}(m)$ and $p_k(O_V)$, respectively.

We often use the following facts to simplify our argument.

Lemma 3.6 *Let $V \subseteq W \subseteq P = \mathbb{P}^N(\mathbb{C})$ be closed subschemes which are both of dimension n, equidimensional, and do not have any non-isolated associated point (e.g. the both schemes are equidimensional and locally Cohen-Macaulay). Assume that $deg(V) = deg(W) = d$. Then we have $V = W$ as schemes.*

Proof. It is enough to show the coincidence of the sheaves of the defining ideals $I_V = I_W$, or equivalently, $\Gamma(D_+(Z_k), I_V) = \Gamma(D_+(Z_k), I_W)$ on any affine open set $D_+(Z_k) = Spec(A_k) \subseteq P$ $(k = 0, 1, ..., N)$, where the ring A_k denotes $\mathbb{C}[(Z_0/Z_k), ..., (Z_N/Z_k)]$. Consider an exact sequence of structure sheaves :

$$0 \longrightarrow I_{V/W} \longrightarrow O_W \longrightarrow O_V \longrightarrow 0.$$

Then, the Hilbert polynomials satisfy that $\chi(I_{V/W}(mH)) = \chi(O_W(mH)) - \chi(O_V(mH)) = \{(d/n!)m^n + (lower\ terms)\} - \{(d/n!)m^n + (lower\ terms)\}$, which implies that the degree of the Hilbert polynomial $\chi(I_{V/W}(mH)))$ is less than or equal to $n-1$, namely $dim(Supp(I_{V/W})) \leq n-1$. Take any irreducible component W_a of the scheme W and its generic point $\zeta_a \in P$. Then, $(I_{V/W})_{\zeta_a} = 0$ and $O_{V,\zeta_a} = O_{W,\zeta_a}$, which means that the stalks of the sheaves of ideals I_{V,ζ_a} and I_{W,ζ_a} are the same $\mathfrak{m}_{P,\zeta_a}$-primary ideals, where $\mathfrak{m}_{P,\zeta_a}$ denotes the unique maximal ideal of the local ring O_{P,ζ_a} at the point ζ_a. If $\zeta_a \in D_+(Z_k)$ or equivalently $W_a \cap D_+(Z_k) \neq \phi$, then the pull-back ideal $\mathfrak{q}_a := \tau_a^{-1}(I_{V,\zeta_a}) = \tau_a^{-1}(I_{W,\zeta_a})$ by the localization map $\tau_a : A_k \to O_{P,\zeta_a}$ is the $\mathfrak{p}_a$-primary component of the ideal $\Gamma(D_+(Z_k), I_V)$ and of the ideal $\Gamma(D_+(Z_k), I_W)$, where the ideal $\mathfrak{p}_a$ denotes the prime ideal corresponding to the generic point ζ_a. Thus we see the coincidence of the primary decompositions of the both ideals $\Gamma(D_+(Z_k), I_V)$ and $\Gamma(D_+(Z_k), I_W)$ because the both ideals do not have any non-isolated associated prime. ∎

By the similar argument, we can easily see the next fact.

Lemma 3.7 *Let $W \subseteq P = \mathbb{P}^N(\mathbb{C})$ be an irreducible (but may not be reduced) closed subscheme. Putting the reduced structure and set a scheme $V := W_{red}$. Take a generic point ζ of W. Then $deg(W) = deg(V) \cdot length(O_{W,\zeta})$.*

Proof. We set $k := length(O_{W,\zeta})$ and use induction argument on k. If we have a non-isolated associated point of W, it does not have an effect on the counting degree, we may remove it by using primary decomposition and assume that the homogeneous ideal $\mathbb{I}_W$ of the scheme W is a primary ideal. If $k = 1$, then $W = V$ and the claim is obvious. Now we assume $k \geq 2$. Then we have an ideal J in the local Artinian ring $O_{W,\zeta}$ which satisfies that $length(O_{W,\zeta}/J) = k - 1$ and $length(J) = 1$. For each affine open set $U = Spec(A)$ of the scheme W, taking the pull-back of the ideal J by the localization map $A \to O_{W,\zeta}$, we obtain a sheaf of ideals $\mathcal{J}$ which satisfies $Supp(\mathcal{J}) = V$ and defines a closed subscheme $Y := (|V|, O_W/\mathcal{J})$ with $length(O_{Y,\zeta}) = k-1$. By induction hypothesis, $deg(Y) = (k-1) \cdot deg(V)$. Since $(I_V \cdot \mathcal{J})_\zeta = 0$, $deg(\mathcal{J}) = deg(\mathcal{J}/I_V \cdot \mathcal{J})$. Then the sheaf $\mathcal{J}/I_V \cdot \mathcal{J}$ is an O_V-module of rank one, which implies $deg(\mathcal{J}/I_V \cdot \mathcal{J}) = deg(V)$. Hence $deg(W) = deg(Y) + deg(\mathcal{J}) = (k-1) \cdot deg(V) + deg(V) = k \cdot deg(V)$. ∎

§4 Classification on PG-shells of a canonical curves X of $g \leq 5$.

Let us take a canonical curves X of genus $g \leq 5$ and a pregeometric shell (scheme) W of X as in Main Theorem (2.1). We classify the scheme W by following the cases below.

• First we consider the case $g = 3$, $g = 4$, or the generic case in $g = 5$, i.e. a non-trigonal case. Then, the canonical curve X is a non-singular quartic curve in $\mathbb{P}^2(\mathbb{C})$, a $(2,3)$-complete intersection in $\mathbb{P}^3(\mathbb{C})$, or a $(2,2,2)$-complete intersection in $\mathbb{P}^4(\mathbb{C})$, respectively (cf. [6] and [7]). Thus, in these cases, we obtain the result as in Main Theorem (2.1) by just applying the claim (3.3.3).

• In the rest of this paper, we consider only the remaining case, namely the canonical curve $X \subseteq P = \mathbb{P}^4(\mathbb{C}) = Proj(S)$ is a trigonal curve of $g = 5$, where the graded ring S denotes the polynomial ring $\mathbb{C}[Z_0, Z_1, Z_2, Z_3, Z_4]$ with the usual grading.

Let us recall the results in [6] and in [7]. The curve X has a unique complete linear system g^1_3 which induces a smooth rational normal scroll surface V of type $S(2,1)$ which satisfies $X \subseteq V \subseteq P$. Since this surface V is a variety of minimal degree, namely a variety of Δ-genus zero, the claim (3.3.9) shows that the surface V is a pregeometric (in fact "geometric") shell of X. The surface V is also defied by all the degree 2 equations of X. This surface V also appeared in our paper [13], which is isomorphic to the one point blow up of a projective plane $Y = \mathbb{P}^2(\mathbb{C})$, namely a rational ruled surface Σ_1. Using the notation of [13], the hyperplane section divisor $H = H|_V$ is linearly equivalent to the divisor $2\xi + E_1$, where the divisor E_1 is the exceptional divisor coming from the one point blow up, and the divisor ξ denotes the strict transform of a line ℓ in $Y = \mathbb{P}^2(\mathbb{C})$ or equivalently a fiber of the morphism $\Sigma_1 \to \mathbb{P}^1(\mathbb{C})$ determined as the rational ruled surface or equivalently as the rational normal scroll surface. As a divisor on the surface V, the canonical curve X is linearly equivalent to the divisor $5\xi + 3E_1 = 3H - \xi$. The homogeneous coordinate ring R_X of the canonical curve X is an arithmetically Gorenstein ring. A minimal S-free resolution $\mathbb{F}_{X\bullet}$ of the ring R_X and a minimal S-free resolution $\mathbb{F}_{V\bullet}$ of the homogeneous coordinate ring R_V of the surface V are given as follows.

(#-1)

$$\begin{array}{ccccccccc}
\mathbb{F}_{X\bullet}: & S & \xleftarrow{\psi_1} & S(-2)^{\oplus 3} \oplus S(-3)^{\oplus 2} & \xleftarrow{\psi_2} & S(-3)^{\oplus 2} \oplus S(-4)^{\oplus 3} & \xleftarrow{\psi_3} & S(-6) & \longleftarrow 0 \\
 & \| & & \uparrow & & \uparrow & & & \\
\mathbb{F}_{V\bullet}: & S & \xleftarrow{\psi_1'} & S(-2)^{\oplus 3} & \xleftarrow{\psi_2'} & S(-3)^{\oplus 2} & \longleftarrow & 0 &
\end{array}$$

Now we consider a minimal S-free resolution $\mathbb{F}_{W\bullet}$ of the homogeneous coordinate ring R_W of the scheme W. Since the scheme W is a pregeometric shell of the curve X, the resolution $\mathbb{F}_{W\bullet}$ has the following form.

(#-2)

$$\mathbb{F}_{W\bullet}: \quad S \xleftarrow{\varphi_1} S(-2)^{\oplus a_1} \oplus S(-3)^{\oplus b_1} \xleftarrow{\varphi_2} S(-3)^{\oplus a_2} \oplus S(-4)^{\oplus b_2} \xleftarrow{\varphi_3} S(-6)^{\oplus a_3} \longleftarrow 0,$$

where $a_1 = 0,1,2,3$; $b_1 = 0,1,2$; $a_2 = 0,1,2$; $b_2 = 0,1,2,3$; and $a_3 = 0,1$. The Hilbert polynomial $A_W(m) = \chi(O_W(mH))$ of the scheme W is calculated by $A_W(m) = A_4(m) - \{a_1A_4(m-2) + b_1A_4(m-3)\} + \{a_2A_4(m-3) + b_2A_4(m-4)\} - a_3A_4(m-6)$. Applying Lemma 3.5, the coefficients $p_k(W)$ of the Hilbert polynomial $A_W(m) = \sum_{k=0}^{4} p_k(W)A_k(m)$ are given by

(#-3)
$$\begin{cases} p_4(W) &= 1 - a_1 + (a_2 - b_1) + b_2 - a_3 \\ p_3(W) &= 2a_1 - 3(a_2 - b_1) - 4b_2 + 6a_3 \\ p_2(W) &= -a_1 + 3(a_2 - b_1) + 6b_2 - 15a_3 \\ p_1(W) &= b_1 - a_2 - 4b_2 + 20a_3 \\ p_0(W) &= b_2 - 15a_3. \end{cases}$$

• • The case of $codim(W) = 1$.

Apply the claims (3.3.1) and (3.3.2). Then we see that the scheme W is a irreducible and reduced hypersurface which corresponds to a memmber of minimal generators for the homogeneous ideal $\mathbb{I}_W$, and there for $degW = 2, 3$, namely its Δ-genus is 0 or 1.

• • The case of $codim(W) = 3$.

Since we assume $codim(W) = 3$, we see that the coefficients of the Hilbert polynomial $A_W(m)$ of W satisfies $p_4(W) = p_3(W) = p_2(W) = 0$. Thus the formula (#-3) shows $a_1 = b_2 = 3$; $a_3 = 1$ and $a_2 = b_1 = 0,1,2$. Then, we have $A_W(m) = 8A_1(m) - 12$ and therefore $A_W(m) = A_X(m)$. Since the curve X is a closed subscheme of the scheme W, we have an exact sequence of sheaves $0 \to I_{X/W} \to O_W \to O_X \to 0$, we have $\chi(I_{X/W}(m)) = A_W(m) - A_X(m) = 0$, which implies $I_{X/W} = 0$, or equivalently $X = W$ as schemes.

• • The case of $codim(W) = 2$.

First we consider the case $a_3 = 1$. Take a homomorphism of complexes $\alpha_\bullet : \mathbb{F}_{W\bullet} \to \mathbb{F}_{X\bullet}$ which arises from the natural ring homomorphism $R_W \to R_X$. Since the homomorphism $\alpha_\bullet$ induces injective homomorphisms $\mu_q : Tor_q^S(R_W, S/S_+) \to Tor_q^S(R_X, S/S_+)$ for $q \geq 0$, in the following diagram, the vertical arrow α_3 is an isomorphism and the vertical arrow α_2 sends the free module $S(-3)^{\oplus a_2} \oplus S(-4)^{\oplus b_2}$ to a direct summand of the free module $S(-3)^{\oplus 2} \oplus S(-4)^{\oplus 3}$.

$$\begin{array}{ccccccc}
\mathbb{F}_{X\bullet}: & S(-3)^{\oplus 2} \oplus S(-4)^{\oplus 3} & \xleftarrow{\psi_3} & S(-6) & \longleftarrow & 0 \\
& \alpha_2 \uparrow & & \uparrow \alpha_3 & & \\
\mathbb{F}_{W\bullet}: & S(-3)^{\oplus a_2} \oplus S(-4)^{\oplus b_2} & \xleftarrow{\varphi_3} & S(-6) & \longleftarrow & 0
\end{array}$$

Now we take a system of (homogeneous) minimal generators $\{G_1, G_2, G_3, F_1, F_2\}$ of the homogeneous ideals $\mathbb{I}_X$ of the curve X with $degG_1 = degG_2 = degG_3 = 2$ and $degF_1 = degF_2 = 3$. The first differential map ψ_1 of the complex $\mathbb{F}_{X\bullet}$ is given by a 1×5-matrix $M_1 = [G_1, G_2, G_3, F_1, F_2]$. Since the homogeneous coordinate ring R_X is an arithmetically Gorenstein ring, the minimal graded S-free resolution $\mathbb{F}_{X\bullet}$ is symmetric, and therefore the 5×1-matrix M_3 of the third differential map ψ_3 of the complex $\mathbb{F}_{X\bullet}$ coincides with the transposed matrix of the matrix $[F_1, F_2, G_1, G_2, G_3]$ up to the (left) action of a graded automorphism of the free module $S(-3)^{\oplus 2} \oplus S(-4)^{\oplus 3}$, which is given by a 5×5-matrix of the form :

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{bmatrix},$$

where $T_{1,1} \in GL(2, \mathbb{C})$, $T_{2,2} \in GL(3, \mathbb{C})$, and $T_{1,2}$ is a 2×3-matrix whose coefficients are all homogeneous polynomials of degree 1. Once we have $a_2 < 2$ or $b_2 < 3$, then we see that one element of the set $\{G_1, G_2, G_3, F_1, F_2\}$ is generated by others, namely $\{G_1, G_2, G_3, F_1, F_2\}$ is not minimal generators of $\mathbb{I}_X$, which contradicts the assumption. Thus we see that $a_2 = 2$ and $b_2 = 3$. On the other hand, from $p_4(W) = 0$ and the formula (#-3), we get $a_1 + b_1 = 5$, which shows $a_1 = 3$ and $b_1 = 2$, namely $X = W$ as schemes. This is a contradiction.

Hence we have $a_3 = 0$. Then, from the resolution (#-2), we see that the homological dimension $hd_S(R_W)$ of the homogeneous coordinate ring R_W satisfies $hd_S(R_W) \leq 2$, namely the depth at the vertex satisfies $arith.depth(R_W) = dimS - hd_S(R_W) \geq 3 = dim(R_W)$ by the Auslander-Buchsbaum formula, which implies that the ring R_W is an arithmetically Cohen-Macaulay ring, $H^0(P, O_P(1)) \cong H^0(W, O_W(1))$, and that the scheme W is locally Cohen-Macaulay and equi-dimensional.

Now we use the assumption $codim(W) = 2$, namely $p_4(W) = p_3(W) = 0$. Applying the formula (#-3) and the fact $a_3 = 0$, we have the 5 cases as in the following list.

Cases	(a_1, b_1)	(a_2, b_2)	a_3
(i)	$(1, 2)$	$(0, 2)$	0
(ii)	$(2, 0)$	$(0, 1)$	0
(iii)	$(2, 1)$	$(1, 1)$	0
(iv)	$(2, 2)$	$(2, 1)$	0
(v)	$(3, 0)$	$(2, 0)$	0

Table 1: Betti Numbers

Now let us recall the resolutions (#-1) and the fact that all the degree 2 equations of the curve X come from those of the surface V of minimal degree. Thus we see that the case (v) means $W = V$ and $\Delta(W, O_W(1)) = 0 \leq \Delta(X, O_X(1)) = 4$.

• • • The cases (ii), (iii), (iv)

In these case, by calculating $p_2(W)$ from the formula (#-3), we have $deg(W) = 4$. On the other hand, the scheme W has independent irreducible degree 2 equations $\{\widetilde{G_1}, \widetilde{G_2}\}$ which come from those of the surface V. Now we take a new closed subscheme $W_0 := \{\widetilde{G_1} = \widetilde{G_2} = 0\}$, which is a $(2,2)$-complete intersection and of degree 4. Since there is an inclusion $W \subseteq W_0$ of arithmetically Cohen-Macaulay schemes with $deg(W) = deg(W_0) = 4$, which means that $W = W_0$ by Lemma 3.6, only the case (ii) is remained.

However, in the case (ii), we have an inclusion of closed schemes $X \subseteq V \subseteq W = W_0$, which induces a graded S/S_+-linear homomorphisms $(S/S_+)_{(4)} \cong Tor_2^S(R_W, S/S_+) \to Tor_2^S(R_V, S/S_+) \cong (S/S_+)_{(3)}^{\oplus 2} \to Tor_2^S(R_X, S/S_+)$, which implies that the induced homomorphism $\mu_2 : Tor_2^S(R_W, S/S_+) \to Tor_2^S(R_X, S/S_+)$ is a zero map and not injective. Thus we obtain a contradiction.

• • • The cases (i)

By the formula (#-3), we have the Hilbert polynomial $A_W(mH) = 5A_2(m) - 6A_1(m) + 2$ of the scheme W, which implies $deg(W) = 5$. By using the fact that the scheme W is arithmetic Cohen-Macaulay and contains the curve X which is not linearly degenerate, we see that $h^0(W, O_W(1)) = 5$. So, in the sense of generalized Δ-genus as in Remark 1.2, we have $\Delta(W, O_W(1)) = 2 + 5 - 5 = 2 \leq \Delta(X, O_X(1)) = 4$.

Since $X \subseteq W$ and the scheme W is equidimensional, there is an irreducible component W_0 of dimension 2 in W with $X \subseteq W_0$. If $deg((W_0)_{red}) \leq 2$, then there is a hyperplane H_0 of P which contains the reduced scheme $(W_0)_{red}$, and therefore the curve X is linearly degenerate, which is a contardiction. Thus we see that $deg((W_0)_{red}) \geq 3$. Since the scheme W is locally Cohen-Macaulay, the scheme W does not have any non-isolated associated point, the scheme W_0 has no embedded point eihter. Take the generic point ζ_0 of the scheme W_0. If $lengthO_{W_0,\zeta_0} \geq 2$, then $deg(W_0) \geq 2deg((W_0)_{red}) \geq 2 \cdot 3 = 6$ by Lemma 3.7, which give a contradiction to the fact $W_0 \subseteq W$. Hence we have $lengthO_{W_0,\zeta_0} = 1$ and the scheme W_0 has no nilpotent structure and is a variety of degree ≥ 3.

• • • • Cases (i-a) : cases (i) & $deg(W_0) = 5$

Now we consider the case $deg(W_0) = 5$. Then the scheme W can not have any other irreducible component of dimension 2, which means that $W = W_0$ and the scheme W itself is also a variety of degree 5. In this case, $\Delta(W, O_W(1)) = 2 \leq \Delta(X, O_X(1)) = 4$.

Remark 4.1 *In the first place, the author expected that the case (i-a) can not happen really. However, Prof. A. Ohbuchi kindly suggested a possibility of the case (i-a) by constructing surfaces of degree 5 containing trigonal curves of genus 5. However, it was not so easy to study the syzygies of these surfaces through this construction. After this suggestion, the author found an example where the case (i-a) really happens by the aid of Lemma 5.1 in the next section. In this example, the curve X never be a hypersurface cut of W because $deg(X) = 8$, $deg(W) = 5$ and $5 \not| 8$. On the other hand, the surface W is of Δ-genus 2, which never be a variety of minimal degree. So, this example is not a typical examples of pregeometric shell already known and is a new type of pregeometric shell. Further discussions on this example will be carried out in the forthcoming paper.*

• • • • Cases (i-b) : cases (i) & $deg(W_0) = 3$

Next we handle the case of $deg(W_0) = 3$. Then the variety W_0 is a variety of Δ-genus ≤ 0 (hence $= 0$) and is defined by the equations of degree 2. Since $X \subseteq W_0$, all of the equations of W_0 come from those of the curve X. Since $deg(W_0) = 3$, it needs at least 3 linearly independent equations of degree 2 to be defined. Thus we see that $W_0 = V$, which means that $(\widetilde{G}, \widetilde{F_1}, \widetilde{F_2})S = \mathbb{I}_W = \mathbb{I}_V \cap \mathbb{I}_{W'} \subseteq \mathbb{I}_V = (G_1, G_2, G_3)S$ where $deg(\widetilde{F_1}) = deg(\widetilde{F_2}) = 3,\ deg(G_1) = deg(G_2) = deg(G_3) = deg(\widetilde{G}) = 2,\ \mathbb{I}_X = (G_1, G_2, G_3, \widetilde{F_1}, \widetilde{F_2})S$ from our previous construction of the scheme W. Then we see that $\widetilde{F_i} \in (G_1, G_2, G_3)S \quad (i = 1, 2)$, which implies that $\{G_1, G_2, G_3, \widetilde{F_1}, \widetilde{F_2}\}$ is not minimal generators of $\mathbb{I}_X$. Thus we obtain a contardiction.

• • • • Cases (i-c) : cases (i) & $deg(W_0) = 4$

In this case, the scheme W is a union of a plane L and the variety W_0 of degree 4 and of dimension 2. Since the surface W_0 includes the curve X, the surface W_0 is not contained by any hyperplane of $P = \mathbb{P}^4$, which implies the inequality $h^0(W_0, O_{W_0}(1)) \geq 5$. Now we consider the Δ-genus of the surface W_0.

$$0 \leq \Delta(W_0, O_{W_0}(1)) = 2 + 4 - h^0(W_0, O_{W_0}(1)) \leq 2 + 4 - 5 = 1$$

First we consider the case $\Delta(W_0, O_{W_0}(1)) = 0$. Then $h^0(W_0, O_{W_0}(1)) = 6$, which means that there exists a surface $\widetilde{W_0}$ in $\widetilde{P} := \mathbb{P}^5 = \mathbb{P}(H^0(W_0, O_{W_0}(1)))$ which is projected isomorphically to the surface W_0 in P by a linear projection corresponding to the natural injective linear map $H^0(P, O_P(1)) \to H^0(W_0, O_{W_0}(1))$. We also have a curve $\widetilde{X}$ in the surface $\widetilde{W_0}$ which is isomorphic to the curve X through this isomorphism from the surface $\widetilde{W_0}$ to the surface W_0. Then we have $h^0(\widetilde{X}, O_{\widetilde{P}}(1) \otimes O_{\widetilde{X}}) = h^0(X, O_X(1)) = 5$, which gives a hyperplane $\widetilde{H}$ of $\widetilde{P}$ satisfying $\widetilde{X} \subseteq \widetilde{H} \cap \widetilde{W_0}$. Thus we obtain $8 = 2 \cdot 5 - 2 = deg(X) = deg(\widetilde{X}) \leq deg\widetilde{W_0} = deg(W_0) = 4$, which is a contradiction.

Next we consider the case $\Delta(W_0, O_{W_0}(1)) = 1$. Let us recall a system of (homogeneous) minimal generators $\{G_1, G_2, G_3, F_1, F_2\}$ of the homogeneous ideals $\mathbb{I}_X$ of the curve X with $degG_1 = degG_2 = degG_3 = 2$ and $degF_1 = degF_2 = 3$. We may assume that the equations $\{G_1, F_1, F_2\}$ give a minimal system of generators of the homogeneous ideal $\mathbb{I}_W$ of the scheme $W = L \cup W_0$. In this case, we have $H^0(W_0, O_{W_0}(1)) \cong H^0(P, O_P(1))$, apply Swinnerton-Dyer's classification on the varieties of degree 4 (cf. [9], [4]), and obtain that the surface W_0 is a $(2,2)$-complete intersection defined by two homogeneous equations of degree 2 : $(\widehat{G_1}, \widehat{G_2})S = \mathbb{I}_{W_0}$. By the fact : $X \subseteq W_0$, we see that $\widehat{G_i} \in \mathbb{I}_X$. Since the equations $\widehat{G_1}$ and $\widehat{G_2}$ are linearly independent and of degree 2, they form a part of a minimal system of generators of $\mathbb{I}_X$. On the other hand, the inclusion $W_0 \subseteq W$ implies $F_i \in \mathbb{I}_{W_0} = (\widehat{G_1}, \widehat{G_2})S$ $(i = 1, 2)$, namely $F_i \in \Gamma(X, I_X(2)) \cdot S_+$. Thus the equations F_i can not be a part of the minimal system of generators of $\mathbb{I}_X$, which gives a contradiction again.

§5 Syzygies of $g = 5$ trigonal curves

In this section, we give a little bit of refinement on minimal free resolutions given in [8] for $g = 5$ trigonal curves. This is also a typical example for the structure theorem in [2] on the minimal free resolutions in the case of codimension 3 and Gorenstein.

Since the trigonal curve X of genus 5 is contained by a unique rational ruled surface V, we first fix the embedding of the surface V in $\mathbb{P}^N(\mathbb{C}) = P$ as follows.

Taking $Y = \mathbb{P}^2(\mathbb{C}) = Proj(\mathbb{C}[U_0, U_1, U_2])$, we get the surface V by blowing up the plane Y at the center $p_0 = [1 : 0 : 0]$. The embedding of V by the linear system $|2\xi + E_1|$, namely the linear system coming from the conics passing through the point p_0, is given by $[Z_0 : Z_1 : Z_2 : Z_3 : Z_4] = [U_0U_1 : U_0U_2 : U_1^2 : U_1U_2 : U_2^2]$. The equations of V are : $G_1 = Z_0Z_3 - Z_1Z_2 = 0$, $G_2 = Z_1Z_3 - Z_0Z_4 = 0$, $G_3 = Z_2Z_4 - Z_3^2 = 0$. When we take a 2×3-matrix :

(#-4)
$$\Phi = \begin{bmatrix} Z_4 & Z_3 & Z_1 \\ Z_3 & Z_2 & Z_0 \end{bmatrix},$$

those equations $G_1,\ G_2,\ G_3$ can be considered as $G_1 = \varphi_{2,3},\ G_2 = -\varphi_{1,3},\ G_3 = \varphi_{1,2}$, where the determinant of 2×2-minor matrix maded by choosing $i,\ j$ columns of Φ is denoted by $\varphi_{i,j}$. The curve X arises from the blowing up at the center p_0 of the plane curve C_0 of degree 5 which is smooth outside the point p_0 and has only one double point at the point p_0. An equation $\widehat{F}_0$ of the plane curve C_0 is written by

(#-5)
$$\widehat{F}_0 = U_0^3\{a_1U_1^2 + a_2U_1U_2 + a_3U_2^2\} + \sum_{3\le e_1+e_2\le 5} c_{e_1,e_2}U_0^{5-e_1-e_2}U_1^{e_1}U_2^{e_2},$$

where $(a_1, a_2, a_3) \neq (0,0,0)$ and the coefficients $a_i,\ c_{e_1,e_2}$ are constants.

Let us consider the following 3 polynomials $\widehat{F}_{0,1},\ \widehat{F}_{0,2}$, and $\widehat{F}_{0,3}$.

(#-6)
$$\begin{aligned} \widehat{F}_{0,1} &= c_{0,5}U_2^4 \\ \widehat{F}_{0,2} &= -\sum_{e_1+e_2=5,\ e_1\ge 1} c_{e_1,e_2}U_1^{e_1-1}U_2^{e_2} \\ \widehat{F}_{0,3} &= U_0^2\{a_1U_1^2 + a_2U_1U_2 + a_3U_2^2\} + \sum_{3\le e_1+e_2\le 4} c_{e_1,e_2}U_0^{4-e_1-e_2}U_1^{e_1}U_2^{e_2} \end{aligned}$$

Then we see that $\widehat{F}_0 = U_0\widehat{F}_{0,3} - U_1\widehat{F}_{0,2} + U_2\widehat{F}_{0,1}$. From these 3 homogeneou polynomials of degree 4 ; $\widehat{F}_{0,1},\ \widehat{F}_{0,2},\ \widehat{F}_{0,3}$, we construct 3 homogeneous polynomials $Q_1,\ Q_2,\ Q_3$ of degree 2 with the variables $Z_0, \cdots Z_4$ by replacing $U_0U_1,\ U_0U_2,\ U_1^2,\ U_1U_2,\ U_2^2$ by $Z_0,\ Z_1,\ Z_2,\ Z_3,\ Z_4$, respectively. Namely,

(#-7)
$$\begin{aligned} Q_1(Z_4) &= c_{0,5}Z_4^2 \\ Q_2(Z_2,Z_3,Z_4) &= -(c_{5,0}Z_2^2 + c_{4,1}Z_2Z_3 + c_{3,2}Z_3^2 + c_{2,3}Z_3Z_4 + c_{1,4}Z_4^2) \\ Q_3(Z_0,Z_1,Z_2,Z_3,Z_4) &= a_1Z_0^2 + a_2Z_0Z_1 + a_3Z_1^2 \\ &\quad + c_{3,0}Z_0Z_2 + c_{2,1}Z_0Z_3 + c_{1,2}Z_1Z_3 + c_{0,3}Z_1Z_4 \\ &\qquad + c_{4,0}Z_2^2 + c_{3,1}Z_2Z_3 + c_{2,2}Z_3^2 + c_{1,3}Z_3Z_4 + c_{0,4}Z_4^2. \end{aligned}$$

Corresponding to the polynomials $U_1\widehat{F}_0 = U_1U_2\widehat{F}_{0,1} - U_1^2\widehat{F}_{0,2} + U_0U_1\widehat{F}_{0,3}$ and $-U_2\widehat{F}_0 = -U_2^2\widehat{F}_{0,1} + U_1U_2\widehat{F}_{0,2} - U_0U_2\widehat{F}_{0,3}$, we set 2 homogeneous polynomials F_1 and F_2 of degree 3 with the variables $Z_0, \cdots, Z_4$:

(#-8)
$$\begin{aligned} F_1 &:= Z_3Q_1 - Z_2Q_2 + Z_0Q_3 \\ F_2 &:= -Z_4Q_1 + Z_3Q_2 - Z_1Q_3. \end{aligned}$$

Summarizing these formula by using matrices, we have :

(#-9)
$$[F_1, F_2] = [Q_1, -Q_2, Q_3]\begin{bmatrix} Z_3 & -Z_4 \\ Z_2 & -Z_3 \\ Z_0 & -Z_1 \end{bmatrix}.$$

Multiplying the matrix Φ from the right hand side to to both sides of the equality (#-9) above, we see that :

$$(\#\text{-}10) \qquad [F_1, F_2] \cdot \Phi = [Q_1, -Q_2, Q_3] \begin{bmatrix} 0 & -\varphi_{1,2} & -\varphi_{1,3} \\ \varphi_{1,2} & 0 & -\varphi_{2,3} \\ \varphi_{1,3} & \varphi_{2,3} & 0 \end{bmatrix}.$$

The key is to transfom the right hand side of the equality (#-10) to the following.

$$(\#\text{-}11) \qquad [F_1, F_2] \cdot \Phi = [\varphi_{2,3}, -\varphi_{1,3}, \varphi_{1,2}] \begin{bmatrix} 0 & Q_3 & Q_2 \\ -Q_3 & 0 & Q_1 \\ -Q_2 & -Q_1 & 0 \end{bmatrix}.$$

To simplify the notation, we set

$$(\#\text{-}12) \qquad M := [F_1, F_2], \quad \Psi = \begin{bmatrix} \varphi_{2,3} \\ -\varphi_{1,3} \\ \varphi_{1,2} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & Q_3 & Q_2 \\ -Q_3 & 0 & Q_1 \\ -Q_2 & -Q_1 & 0 \end{bmatrix}.$$

Then the equality (#-11) and its transposed formula are written as follows.

$$(\#\text{-}13) \qquad \begin{array}{rcl} M \cdot \Phi & = & {}^t\Psi \cdot Q \\ {}^t\Phi \cdot (-{}^tM) & = & Q \cdot \Psi \end{array}$$

Now, using the formula (#-13), we shall construct a minimal graded S-free resolution of the homogeneous coordinate ring R_X as prescribed by Schreyer in [8] with a little bit of refinement. Let us remind that $X = (\sigma_X)_0$ by a section $\sigma_X \in H^0(V, O_V(3H - \xi))$. Hence we have an exact sequence :

$$(\#\text{-}14) \qquad 0 \longrightarrow O_V(-3H + \xi) \xrightarrow{\times \sigma_X} O_V \longrightarrow O_X \longrightarrow 0.$$

Then we recall the 2×3-matrix Φ in (#-4), which can be considered as a sheaf homomorphism $\Phi : \mathcal{F} = \oplus^3 O_P(-H) \to \mathcal{G} = \oplus^2 O_P$. By the technique of [1], we can obtain a family of O_P-(locally) free complexes $\{\mathcal{C}^b_\bullet\}_{b \geq -1}$. Here, the Eagon-Northcott complex $\mathcal{C}^0_\bullet$ and the Buchsbaum-Rim complex $\mathcal{C}^1_\bullet$ give minimal O_P-(locally) free resolutions for O_V and $O_V(\xi)$, respectively. Recall our notation (#-4) and (#-12). Then,

(#-15)

$$\mathcal{C}^0_\bullet : \ [\, 0 \longleftarrow O_V \longleftarrow \,]\, O_P \xleftarrow{{}^t\Psi} O_P(-2H)^{\oplus 3} \xleftarrow{-{}^t\Phi} O_P(-3H)^{\oplus 2} \longleftarrow 0$$

$$\mathcal{C}^1_\bullet : \ [\, 0 \longleftarrow O_V(\xi) \longleftarrow \,]\, O_P^{\oplus 2} \xleftarrow[\Phi]{} O_P(-H)^{\oplus 3} \xleftarrow[\Psi]{} O_P(-3H) \longleftarrow 0.$$

Using our formula (#-13), we lift the sheaf homomorphism $\sigma_X : O_V(-3H + \xi) \to O_V$ to a complex homomorphism $\alpha_\bullet : \mathcal{C}^1_\bullet(-3H) \to \mathcal{C}^0_\bullet$.

$$
(\#\text{-}16)\qquad
\begin{array}{cccccccc}
\mathcal{C}^0_\bullet : & O_P & \xleftarrow{{}^t\Psi} & O_P(-2H)^{\oplus 3} & \xleftarrow{{}^t\Phi} & O_P(-3H)^{\oplus 2} & \longleftarrow & 0 \\
 & \uparrow{\scriptstyle \alpha_0 = M} & & \uparrow{\scriptstyle \alpha_1 = Q} & & \uparrow{\scriptstyle \alpha_2 = -{}^tM} & & \\
\mathcal{C}^1_\bullet(-3H) : & O_P^{\oplus 2}(-3H) & \xleftarrow[\Phi]{} & O_P(-4H)^{\oplus 3} & \xleftarrow[\Psi]{} & O_P(-6H) & \longleftarrow & 0.
\end{array}
$$

Then the mapping cone $Cone[\alpha_\bullet : \mathcal{C}^1_\bullet(-3H) \to \mathcal{C}^0_\bullet]$ gives a minimal O_P(-locally) free resolution of O_X or of the homogeneous coordinate ring R_X of the curve X.

$$
(\#\text{-}17)\qquad
\begin{array}{l}
0 \longleftarrow R_X \longleftarrow S \xleftarrow{\delta_1} S(-2)^{\oplus 3} \oplus S(-3)^{\oplus 2} \\
\qquad\qquad \xleftarrow{\delta_2} S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2} \xleftarrow{\delta_3} S(-6) \longleftarrow 0,
\end{array}
$$

where each differential map is defined by a matrix :

$$
(\#\text{-}18)\qquad
\begin{aligned}
\delta_1 &= [{}^t\Psi, M] \\
\delta_2 &= \begin{bmatrix} Q & {}^t\Phi \\ -\Phi & 0 \end{bmatrix} \\
\delta_3 &= \begin{bmatrix} -\Psi \\ -{}^tM \end{bmatrix}.
\end{aligned}
$$

These matrices act from the left, and every element in a S-free module is considered as a column vector with S-valued coefficients. Related to the work of [2], we make a remark that the middle differential map δ_2 is represented by a skew symmetric matrix.

Using the fact above, let us consider again the case (i) in the previous section.

Lemma 5.1 *Let $X \subseteq \mathbb{P}^4(\mathbb{C}) = P$ be a trigonal curve of genus 5. Then, in the following three conditions, each former condition implies the latter condition.*

(5.1.1) *There exists a pregeometric shell (scheme) W of X as in the case (i), namely its dimension is 2 and the Betti numbers $(a_1, b_1, a_2, b_2, a_3) = (1, 2, 0, 2, 0)$.*

(5.1.2) *In the resolution (#-17), there are two surjective S-linear homomorphisms $p_1 : S(-2)^{\oplus 3} \oplus S(-3)^{\oplus 2} \to S(-2)^{\oplus 2}$ and $p_2 : S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2} \to S(-4) \oplus S(-3)^{\oplus 2}$ such that the homomorphism $p_1 \circ (\delta_2|_K) : K := Ker(p_2) \to S(-2)^{\oplus 2}$ is a zero map.*

(5.1.3) *We can find suitable two 2×2-matrices H, H' with coefficients in homogeneous polynomials of S in degree 1, and 2×3-matrix T, and 3×2 matrix T' whose rank are 2 and both coefficients are constants such that*

$$T \cdot Q \cdot T' = H \cdot \Phi \cdot T' - T \cdot ({}^t\Phi) \cdot H'.$$

Proof. First we show that the condition (5.1.1) implies the condition (5.1.2). From [17], we have an exact commutative diagram:

(#-19)

$$\begin{tikzcd}
 & & & S(-2)^{\oplus 2} & \begin{matrix} S(-4) \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} & & \\
0 & R_X \arrow[l] & S \arrow[l] & \begin{matrix} S(-2)^{\oplus 3} \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta_1"'] \arrow[u, "p_1"] & \begin{matrix} S(-4)^{\oplus 3} \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta_2"'] \arrow[u, "p_2"'] & S(-6) \arrow[l, "\delta_3"'] & 0 \arrow[l] \\
0 & R_W \arrow[l] \arrow[u] & S \arrow[l] \arrow[u, equal] & \begin{matrix} S(-2) \\ \oplus \\ S(-3)^{\oplus 2} \end{matrix} \arrow[l, "\delta_1'"'] \arrow[u, "i_1"] & S(-4)^{\oplus 2} \arrow[l, "\delta_2'"'] \arrow[u, "i_2"'] & 0, \arrow[l] &
\end{tikzcd}$$

where the homomorphisms i_1 and i_2 are induced as a part of an induced homomorphism of chain complexes from the canonical ring homomorphism $R_W \to R_X$, and the homomorphisms p_1 and p_2 are defined by canonical quotient maps. Then we see that $p_1 \circ \delta_2 \circ i_2 = 0$ and therefore that the condition (5.1.2) holds.

Next we see that the condition (5.1.2) implies the condition (5.1.3). Obviously, $K = Ker(p_2) \cong S(-4)^{\oplus 2}$, which is also a direct summand of the module $S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2}$. Let us denote the inclusion homomorphism of K into $S(-4)^{\oplus 3} \oplus S(-3)^{\oplus 2}$ by i_K. Then the homomorphisms p_1 and i_K are represented by the matrices :

(#-20)
$$\begin{aligned} p_1 &= [T, H] \\ i_K &= \begin{bmatrix} T' \\ H' \end{bmatrix}, \end{aligned}$$

where the matrices T, H, T' and H' satisfy the conditions as in (5.1.3). Then, we just write down the conditons $p_1 \circ \delta_2 \circ i_2 = 0$ by using the formula (#-18) and the matrices in (#-20) and get the equality in the condition (5.1.3). ∎

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