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Pregeometric Shell type Extensions of Graded Modules

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Abstract

By generalizing the definition of Pregeometric Shells (cf. [11], [12]), we define a concept " Pregeometric Shell type Extensions " (abbr. PGS-extensions) of finite graded modules over the plynomial rings. We also give several characterizations of PGS-extensions and important examples relating with PGS-extensions.

Keywords: minimal free resolution, pregeometric shell, pregeometric shell type extension (PGS-extension)

§0 Introduction

Our main concern is to study the "geometric structure" of a projective embedding of a given variety X (cf. several fundamental problems in [12]). In other words, it is to see the "pregeometric shells" (abbr. PG-shells...cf. [11], [12]), namely intermediate ambient schemes satisfying certain good conditions from the view point of syzygies for the embedded variety X. This is similar to Galois Theory where we study intermediate fields to see the structure of the given extension of a field.

The concept of pregeometric shells is the one on the subideals of the homogeneous ideal $\mathbb{I}_X$ defining the embedded variety X. However, since all the homogeneous ideals of the plynomial ring $S := \mathbb{C}[Z_0, \dots , Z_N]$ and inclusion maps do not form a good category such as an abelian category, it is better to generalize this concept for the case of graded S-modules which form an abelian category. We can also expect that this generalization brings us convenience for studying syzygy modules.

Thus we give a definition of " Pregeometric Shell type Extensions " (abbr. PGS-extensions) of finitely generated graded modules over the plynomial ring S and present severeal critera for PGS-extensions. We will also show some examples which clarify the limits of these criteria and the difficulty of studying pregeometric shells.

§1 Preliminaries.

First we summarize what will be used throughout this paper.

Notation and Conventions 1.1 *We use the terminology of* [4], [5], [8], [9] *without mentioning so, always admit the conventions, and use the notation below for simplicity.*

(1.1.1) *Every object under consideration is defined over the field of complex numbers $\mathbb{C}$. We will work mainly in the category of finitely generated (abbr. " finite") graded modules over the plynomial ring S and graded S-linear homomorphisms. In this case, an* element *of a graded S-module M*

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or of the ring S always means a homogeneous element of the module M or of the ring S. In some exceptional cases, our consideration is carried out also in the category of algebraic schemes and algebraically holomorphic morphisms, or in the categories of coherent sheaves and their ($\mathcal{O}$-linear) homomorphisms.

(1.1.2) *For a coherent sheaf E on a projective subscheme $V \subseteq P = \mathbb{P}^N(\mathbb{C})$, we put: $\Gamma_*(E) := \bigoplus_{m\in\mathbb{Z}} \Gamma(V, E(m))$. Except a few cases where we have to avoid confusion, we do not use distinguished fonts comprehensively for the graded S-modules and sheaves such as M and $\mathcal{M}$, respectively. If we need to distinguish clearly a sheaf M from the S-module $\Gamma_*(M)$,we use the blackboard bold font for S-module such as $\mathbb{M} = \Gamma_*(M)$. Also, for example, we denote $M_\bullet$ for complex of sheaves and $\mathbb{M}_\bullet$ for complex of S-modules, respectively.*

(1.1.3) *Let us take a complex projective scheme X of dimension n and one of its embeddings $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. The sheaf of ideals defining $j(X)$ in P is denoted by I_X. Take a $\mathbb{C}$-basis $\{Z_0, \dots , Z_N\}$ of $H^0(P, O_P(1))$. Then we set:*

$$
\begin{aligned}
S &:= \mathbb{C}[Z_0, \dots , Z_N] \cong \bigoplus_{m\geq 0} H^0(P, O_P(m)) \\
S_+ &:= (Z_0, \dots , Z_N)S \cong \bigoplus_{m>0} H^0(P, O_P(m)) \\
\widetilde{R_X} &:= \bigoplus_{m\geq 0} H^0(X, O_X(m)) \\
\mathbb{I}_X &:= \bigoplus_{m\geq 0} H^0(P, I_X(m)) \\
R_X &:= Im[S \to \widetilde{R_X}] \cong S/\mathbb{I}_X.
\end{aligned}
\tag{\#-1}
$$

(1.1.4) *For any graded S-module or graded S-linear homomorphism, its induced (S/S_+)-object obtained by tensoring (S/S_+) is always denoted by adding overline on the top of the name of the original object. For example, let L, M be finite graded S-modules, and $\varphi : L \to M$ a graded S-linear homomorphism. Then $\overline{M} := M \otimes (S/S_+)$, and the induced (S/S_+)-linear homomorphism $\varphi \otimes 1_{(S/S_+)} : \overline{L} = L\otimes(S/S_+) \to \overline{M} = M\otimes(S/S_+)$ is denoted by $\overline{\varphi}$. Also the induced homomorphisms for Tor groups or for Ext groups from the homomorphism φ are denoted by φ_* in covariantly induced cases such as $\varphi_* : Tor^S_q(L, S/S_+) \to Tor^S_q(M, S/S_+)$, and φ^* in contravariantly induced cases such as $\varphi^* : Ext^q_S(M, S/S_+) \to Ext^q_S(L, S/S_+)$, respectively.*

We say that $\mathbb{F}_\bullet \to M : \cdots \overset{\mu_{k+1}}{\to} F_k \overset{\mu_k}{\to} \cdots \overset{\mu_1}{\to} F_0 \overset{\mu_0}{\to} M \to 0$ is a minimal *graded S-free resolution of a finite graded S-module M, if it is a graded S-free resolution with $\overline{\mu_k} = 0$ for $k \geq 1$. It is well-known that a minimal graded S-free resolution is unique up to a (non-canonical) complex isomorphism* (cf. [8], [10]).

In the process of constructing a graded S-free resolution $\mathbb{F}_\bullet \to M : \cdots \overset{\mu_{k+1}}{\to} F_k \overset{\mu_k}{\to} \cdots \overset{\mu_1}{\to} F_0 \overset{\mu_0}{\to} M \to 0$ from a finite graded S-module M, to make our argument inductive, the symbol μ_0 denotes always an augmentation homomorphism $\mu_0 : F_0 \to M$ instead of using the traditional symbol ε. After we move to the process of constructing homology objects or cohomology objects such as $Tor^S_*(M, -)$ or $Ext^*_S(M, -)$, without mentioning, we replace the homomorphism μ_0 and the module M by the zero homomorphism and the zero module, respectively, and denote it by $\mathbb{F}_\bullet$ with removing "$\to M$".

(1.1.5) *For a finite graded S-module M, we denote the degree m part of M by $M_{(m)}$, namely $M = \bigoplus_{m\in\mathbb{Z}} M_{(m)}$. To describe the homomorphisms of graded S-free modules clearly, we often descrbe a graded S-free module by $F = \oplus^n_{i=1} Se_i$ with using a free basis $\{e_i | deg(e_i) = m_i\}^n_{i=1}$ instead of* *by* $F = \oplus_{i=1}^{n} S(-m_i)$ *with using degree shifting. In this case, the isomorphism on each direct summand is given by* $S(-m) \supseteq S(-m)_{(k)} = S_{(k-m)} \ni g \leftrightarrow g \cdot e \in (S \cdot e)_{(k)} \subseteq S \cdot e$*, where* $deg(e) = m$.

For a finite graded S*-module* M*, its homogeneous elements* $\{\tau_i \in M_{(m_i)}\}_{i=1}^{n}$*, and for a graded* S*-free module* $L := \oplus_{i=1}^{n} Se_i$*, an* S*-linear homomorphism* $\varphi : L = \oplus_{i=1}^{n} Se_i \to M$ *with* $\varphi(e_i) = \tau_i$ *is denoted simply by* $\varphi : L = \oplus_{i=1}^{n} S[\tau_i] \to M$*, namely* $[\tau_i] = e_i$ *and* $[\tau_i]$ *is a member of the basis of* L *which is sent to the element* τ_i *of the module* M *by the homomorphism* φ*. When we handle a Koszul complexe* $\mathbb{F}_\bullet = \{(F_k, \mu_k)\}_{k \geq 0} \to M = (f_1, \dots f_n)S$ *for an (homogeneous) ideal generated by a (homogeneous)* S*-regular sequence* $\{f_1, \dots f_n\}$*, we also describe its* k*-th differential map* $\mu_k : F_k = \oplus S[f_{i_0}] \wedge \cdots \wedge [f_{i_k}] \to F_{k-1} = \oplus S[f_{j_0}] \wedge \cdots \wedge [f_{j_{k-1}}]$ *as sending the element* $[f_{i_0}] \wedge \cdots \wedge [f_{i_k}]$ *in the basis of* F_k *to the element* $\sum_{t=0}^{k} (-1)^t f_{i_t} [f_{i_0}] \wedge \cdots \overset{t}{\smile} \cdots \wedge [f_{i_k}]$ *of* F_{k-1}.

The following concepts are our main concern to study in this article.

Definition 1.2 *Let us take an exact sequence :*

$$(\#\text{-}2) \qquad 0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0,$$

where M'*,* M *and* M'' *denote finite graded* S*-modules, and the maps* φ *and* ψ *are graded* S*-linear homomorphisms. The exact sequence* (#-2) *is called* a pregeometric shell type extension (abbr. PGS-extension) *if for any non-negative integer* q*, the sequence :*

$$(\#\text{-}3) \quad 0 \longrightarrow Tor_q^S(M', S/S_+) \xrightarrow{\varphi_*} Tor_q^S(M, S/S_+) \xrightarrow{\psi_*} Tor_q^S(M'', S/S_+) \longrightarrow 0,$$

induced from the sequence (#-2) *after tensoring* S/S_+ *is exact. In other words, the long Tor sequence induced from the sequence* (#-2) *breaks into each short exact sequence, namely all the connecting homomorphisms are zero maps.*

In this case, we also say that the module M' *is* a submodule of a pregeometric shell type of M (abbr. PGS-submodule of M) *and the module* M'' *is* a quotient module of a pregeometric shell type of M (abbr. PGS-quotient module of M).

Now we take finite graded S*-modules* M*,* L *and a graded* S*-linear homomorphism* $\varphi : L \to M$*. Then the homomorphism* φ *is called a* monoPGS-homomorphism *if for any non-negative integer* q*, the induced homomorphism* $\varphi_* : Tor_q^S(L, S/S_+) \to Tor_q^S(M, S/S_+)$ *is injective. Similarly, a homomorphism* $\psi : M \to L$ *is called a* epiPGS-homomorphism *if for any non-negative integer* q*, the induced homomorphism* $\psi_* : Tor_q^S(M, S/S_+) \to Tor_q^S(L, S/S_+)$ *is surjective.*

The next claims on monoPGS-homomorphisms and epiPGS-homomorphisms are obvious but often useful.

Lemma 1.3 *Let* M_1*,* M_2*,* M_3 *be finite graded* S*-modules and* $\varphi_1 : M_1 \to M_2$*,* $\varphi_2 : M_2 \to M_3$ *graded* S*-linear homomorphisms. Then, we have the following properties.*

(1.3.1) *If the homomorphisms* φ_1 *and* φ_2 *are monoPGS-homomorphisms, then the composed homomorphism* $\varphi_2 \circ \varphi_1$ *is also a monoPGS-homomorphism. On the other hand, if the composed homomorphism* $\varphi_2 \circ \varphi_1$ *is a monoPGS-homomorphism, then the homomorphism* φ_1 *is also a monoPGS-homomorphism.*

(1.3.2) *If the homomorphisms φ_1 and φ_2 are epiPGS-homomorphisms, then the composed homomorphism $\varphi_2 \circ \varphi_1$ is also an epiPGS-homomorphism. On the other hand, if the composed homomorphism $\varphi_2 \circ \varphi_1$ is an epiPGS-homomorphism, then the homomorphism φ_2 is also an epiPGS-homomorphism.*

(1.3.3) *If the short exact sequence: $0 \to M_1 \overset{\varphi_1}{\to} M_2 \overset{\varphi_2}{\to} M_3 \to 0$ splits, then the homomorphism φ_1 is a monoPGS-homomorphism and the homomorphism φ_2 is an epiPGS-homomorphism.*

Remark 1.4 *In spite of the claim* (1.3.3)*, there are many important non-splitting PGS-extensions in our applications. One of the aims of this article is to clarify homologically hidden splitting structures of PGS-extensions* (cf. Theorem 2.6 (2.6.2)).

Let us see the relation among epiPGS-homomorphisms, monoPGS-homomorphisms and PGS-extensions.

Lemma 1.5 *An epiPGS-homomorphism $\psi : M \to L$ is always surjective and induces a PGS-extension:*

(#-4)
$$0 \longrightarrow Ker(\psi) \longrightarrow M \xrightarrow{\psi} L \longrightarrow 0.$$

Proof. Since in case of $q = 0$, the surjective condition on $Tor_q^S(M, S/S_+) \to Tor_q^S(L, S/S_+)$ implies the surjectivity of the induced map $\overline{\psi} : M \otimes (S/S_+) \to L \otimes (S/S_+)$, which and Nakayama's Lemma imply the surjectivity of ψ. ∎

Remark 1.6 *On the other hand, a monoPGS-homomorphism $\varphi : L \to M$ is not always injective. For example, we put $M = S/S_+$, $L = S$ and set the homomorphism φ to be the canonical quotient map $\varphi : L = S \to M = S/S_+$, which is obviously not injective. Then, it is easy to check the injectivities of $Tor_q^S(L, S/S_+) \to Tor_q^S(M, S/S_+)$ for any non-negative integer q.*

Here we should make a remark that there exists a surjective and non-isomorphic monoPGS-homomorphism as the example above.

One of the reason why we admit non-injective monoPGS-homomorphisms is explained in Lemma

However, once we obtain a monoPGS-homomorphism, there is a canonical way to construct a PGS-extension from this homomorphism which preserve almost all information on the Tor-groups and the induced homomorphisms.

Lemma 1.7 *Assume that a monoPGS-homomorphism $\varphi : L \to M$ is given. Then, there exists a PGS-extension :*

(#-5)
$$0 \longrightarrow K' \xrightarrow{\Phi} K \longrightarrow K/K' \longrightarrow 0$$

such that for any non-negative integer q, there exist isomorphisms $\delta' : Tor_{q+1}^S(L, S/S_+) \overset{\sim}{\to} Tor_q^S(K', S/S_+)$ and $\delta : Tor_{q+1}^S(M, S/S_+) \overset{\sim}{\to} Tor_q^S(K, S/S_+)$, which bring us a compatibility of the induced homomorphisms φ_* and Φ_* :

(#-6)
$$\begin{array}{ccc} Tor_q^S(K', S/S_+) & \xrightarrow{\Phi_*} & Tor_q^S(K, S/S_+) \\ \cong \delta' \uparrow & & \uparrow \delta \cong \\ Tor_{q+1}^S(L, S/S_+) & \xrightarrow[\varphi_*]{} & Tor_{q+1}^S(M, S/S_+). \end{array}$$

Moreover, if the monoPGS-homomorphism $\varphi : L \to M$ *is surjective, we obtain a short exact sequence* $0 \to K/K' \to L \to M \to 0$ *which induces naturally a short exact sequence:*

(#-7)
$$0 \longrightarrow Tor_{q+1}^S(L, S/S_+) \xrightarrow{\varphi_*} Tor_{q+1}^S(M, S/S_+) \xrightarrow{\delta''} Tor_q^S(K/K', S/S_+) \longrightarrow 0.$$

On the other hand, if the monoPGS-homomorphism $\varphi : L \to M$ *is injective, obviously the sequence* $0 \to L \xrightarrow{\varphi} M \to M/L \to 0$ *itself is a PGS-extension.*

Proof. By the definition, for any non-negative integer q, the induced homomorphism $\varphi_* : Tor_q^S(L, S/S_+) \to Tor_q^S(M, S/S_+)$ is injective. In particular, the homomorphism $\overline{\varphi} : L \otimes (S/S_+) \to M \otimes (S/S_+)$ is injective, which means that a system of minimal generators of L forms a part of a system of minimal generators of M. Corresponding to those systems of minimal generators, we can take graded S-free modules F' and F with the surjective graded S-linear homomorphisms $p' : F' \to L$ and $p : F \to M$ which induce isomorphisms $\overline{p'} : F' \otimes (S/S_+) \xrightarrow{\sim} L \otimes (S/S_+)$ and $\overline{p} : F \otimes (S/S_+) \xrightarrow{\sim} M \otimes (S/S_+)$, respectively.

Then, using the surjectivity of the homomorphism $p : F \to M$ and the projectivity of the graded S-free module F', we obtain a graded S-linear lift $\Phi : F' \to F$ of the homomorphism $\varphi \circ p' : F' \to M$, namely $p \circ \Phi = \varphi \circ p'$.

Since the monoPGS-homomorphism $\varphi : L \to M$ induces an injective S/S_+-linear homomorphism $\overline{\varphi} : L \otimes (S/S_+) \to M \otimes (S/S_+)$, we see that $\overline{\Phi} : F' \otimes (S/S_+) \to F \otimes (S/S_+)$ is also injective. Let us represent the homomorphism Φ and the homomorphism $\overline{\Phi}$ by matrices A and $\overline{A}$ whose entries belong to the polynomial ring S and to the residue field S/S_+, respectively. Since the matrix A goes to the matrix $\overline{A}$ via the canonical map $S \to S/S+$, by considering determinants of minor square matrices of those matrices, we see that $rank_S(F') \geq rank(A) \geq rank(\overline{A}) = rank_{S/S_+}(F' \otimes (S/S_+))$, which implies that all the inequalities are the equalities. Thus the homomorphim $\Phi : F' \to F$ is also injective. Moreover, the injectivity of the homomorphism $\overline{\Phi}$ shows that $Tor_1^S(F/F', S/S_+) = 0$, which means that finite graded S-module $F'' := F/F'$ is also S-free and therefore S-free submodule F' is a direct summand of F. We put $K := Ker[p : F \to M]$ and $K' := Ker[p' : F' \to L]$. Then the homomorphism $\Phi : F' \to F$ naturally induces a homomorphism $\Phi : K' \to K$ and brings us an exact sequence (#-5), which is included in the following exact commutative diagram:

(#-8)
$$
\begin{tikzcd}
 & 0 & 0 & & \\
 & L \arrow[u] \arrow[r, "\varphi"] & M \arrow[u] & & \\
0 \arrow[r] & F' \arrow[u, "p'"] \arrow[r, "\Phi"] & F \arrow[u, "p"'] \arrow[r] & F'' \arrow[r] & 0 \\
0 \arrow[r] & K' \arrow[u] \arrow[r, "\Phi"'] & K \arrow[u] \arrow[r] & K/K' \arrow[u] \arrow[r] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

Now we apply $Tor^S_*(-, S/S_+)$ to the diagram (#-8) and get the compatibility (#-7), where the isomorphisms δ' and δ are coming from the connecting homomorphisms of Tor-groups.

If the monoPGS-homomorphism φ is surjective, the free module F'' in the diagram (#-8) is zero, which induces an exact sequence $0 \to K/K' \to L \to M \to 0$ by Snake Lemma. ∎

As the corollary of the proof of Lemma 1.7 above, we obtain a claim as follows.

Corollary 1.8 *Let $\varphi : L \to M$ be a graded S-linear homomorphism whose induced homomorphism $\overline{\varphi} : L \otimes (S/S_+) \to M \otimes (S/S_+)$ is injective, $\mu' : F' \to L$ and $\mu : F \to M$ graded S-linear homomorphisms from graded S-free modules F' and F which induce isomorphisms $\overline{\mu'} : F' \otimes (S/S_+) \xrightarrow{\sim} L \otimes (S/S_+)$ and $\overline{\mu} : F \otimes (S/S_+) \xrightarrow{\sim} M \otimes (S/S_+)$, respectively. Take any S-linear lift $\Phi : F' \to F$ of $\varphi : L \to M$, namely $\mu \circ \Phi = \varphi \circ \mu'$. Then F' is a direct summand of F via Φ.*

Lemma 1.9 *Let us take finite graded S-modules M, L and a graded S-linear homomorphism $\varphi : L \to M$. Then, the following three conditions are equivalent.*

(1.9.1) *The homomorphism φ is an isomorphism.*

(1.9.2) *The homomorphism φ is a monoPGS-homomorphism and also an epiPGS-homomorphism.*

(1.9.3) *The induced homomorphisms $\varphi_* : Tor^S_q(L, S/S_+) \to Tor^S_q(M, S/S_+)$ are isomorphic for $q = 0$ and $q = 1$.*

Proof. The implications (1.9.1) $\Rightarrow$ (1.9.2) $\Rightarrow$ (1.9.3) are obvious. To show the implication (1.9.3) $\Rightarrow$ (1.9.1), we take partial minimal graded S-free resolutions $F'_1 \to F'_0 \to L \to 0$ and $F_1 \to F_0 \to M \to 0$, respectively. Then, from the homomorphism $\varphi : L \to M$, we obtain S-linear lifts $\Phi_0 : F'_0 \to F_0$ and $\Phi_1 : F'_1 \to F_1$, which bring us an exact commutative diagram :

(#-9)
$$
\begin{tikzcd}
F'_1 \arrow[r] \arrow[d, "\Phi_1"'] & F'_0 \arrow[r] \arrow[d, "\Phi_0"'] & L \arrow[r] \arrow[d, "\varphi"] & 0 \\
F_1 \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}
$$

The condition (1.9.3) is the same to say that the homomorphisms $\overline{\Phi_q} : F'_q \otimes (S/S_+) \to F_q \otimes (S/S_+)$ are isomorphic for $q = 0,\ 1$. Using similar argument in the proof of Lemma 1.7, we see that the homomorphisms Φ_q are also isomorphic for $q = 0,\ 1$. Then, diagram chasing in the diagram (#-9) shows that the homomorphism φ is isomorphic, namely the condition (1.9.1). ∎

Remark 1.10 *In Lemma 1.7, the induced PGS-extension $0 \to K' \to K \to K/K' \to 0$ loses only the information on the part $Tor_0^S(L, S/S_+) \to Tor_0^S(M, S/S_+)$. One might wonder whether or not there exists a PGS-extension $0 \to L' \overset{\psi}{\to} M' \to M'/L' \to 0$ with (S/S_+)-isomorphisms $\alpha_q : Tor_q^S(L, S/S_+) \overset{\sim}{\to} Tor_q^S(L', S/S_+)$ and $\beta_q : Tor_q^S(M, S/S_+) \overset{\sim}{\to} Tor_q^S(M', S/S_+)$ such that the induced homomorphisms ψ_* satisfy $\psi_* \circ \alpha_q = \beta_q \circ \varphi_*$.*

If we assume that those (S/S_+)-isomorphisms $\{\alpha_q\}_q$ and $\{\beta_q\}_q$ are the induced homomorphisms of the certain S-linear homomorphisms $\alpha : L \to L'$ and $\beta : M \to M'$ with $\beta \circ \varphi = \psi \circ \alpha$, then, by Lemma 1.9, the S-linear homomorphisms α and β are isomorphic, which implies obviously the non-existece of such a PGS-extension.

In case we assume only the existence of those (S/S_+)-isomorphisms $\{\alpha_q\}_q$ and $\{\beta_q\}_q$, we have to consider more carefully for showing the non-existece of such a PGS-extension. Let us recall the counter-example in Remark 1.6. Since $L = S$, $M = S/S_+$, and $\varphi : L = S \to M = S/S_+$ is the canonical quotient map, we see that $Tor_0^S(L', S/S_+) \cong S/S_+$, $Tor_1^S(L', S/S_+) = 0$, $Tor_0^S(M', S/S_+) \cong S/S_+$, and $Tor_{N+1}^S(M', S/S_+) \cong S/S_+$. Then the S-module L' is generated by one element and S-free, namely $L' \cong S$. Moreover, the S-module M' is also generated by one element and of homological dimension $N+1$, or equivalently of depth zero. Thus we see that $M' \cong S/I$ and the maximal ideal S_+ is an associated prime of the ideal I. On the other hand, if the S-linear homomorphism $\psi : L' \cong S \to M' \cong S/I$ is injective, the zero ideal (0) is an associated prime of the ideal I, which means $I = (0)$, which never has the maximal ideal S_+ as an associated prime. Thus we get a contradiction.

Let us see typical and important two examples (one of them is given as a lemma) where PGS-extensions or monoPGS-homomorphisms appear.

Example 1.11 *Let us take a complex projective subscheme $X \subseteq P = \mathbb{P}^N(\mathbb{C})$ of dimension $n > 0$. Then, for any intermediate closed subscheme W, namely $X \subseteq W \subseteq P$, the scheme W is a PG-shell of X (cf. [11], [12]) if and only if the sequence : $0 \to \mathbb{I}_W \to \mathbb{I}_X \to \mathbb{I}_X/\mathbb{I}_W \to 0$ is a PGS-extension. In particular, if the subscheme X is non-degenerate and the scheme W is a variety of minimal degree, then the scheme W is a PG-shell of X.*

Lemma 1.12 *Let M be a finite graded S-module and $\{f_1, \dots f_k\}$ a (homogeneous) M-regular sequence. Then, the canonical quotient homomorphism $\varphi : M \to M/\Sigma_{i=1}^k f_i M$ is a monoPGS-homomorphism.*

Proof. We just imitate the proof of Lemma 1.8 in [11]. By Lemma 1.7, we may assume $k = 1$ and set $f := f_1$. From the short exact sequence :

(#-10) $$0 \longrightarrow M \xrightarrow{\times f} M \xrightarrow{\varphi} M/fM \longrightarrow 0,$$

we obtain an exact sequence

(#-11) $$Tor_q^S(M, S/S_+) \xrightarrow{\times f} Tor_q^S(M, S/S_+) \xrightarrow{\varphi_*} Tor_q^S(M/fM, S/S_+).$$

Since the Tor group $Tor_*^S(-, S/S_+)$ is a S/S_+-module and is anihilated by the element $f \in S_+$, we obtain what we wanted. ∎

Corollary 1.13 *Let $X \subseteq W \subseteq P = \mathbb{P}^N(\mathbb{C})$ be closed subschemes. Assume that the subscheme W is arithmetically Cohen-Macaulay and the sequence $\{F_1, \dots, F_k | F_i \in H^0(P, O_P(m_i))\}$ is a O_W-regular sequence. Set the sheaf of ideal $I_{Y/W}$ on the scheme W to be*

(#-12) $$I_{Y/W} := Im\ [\ \bigoplus_{i=1}^k O_W(-m_i) \xrightarrow{(F_1,\dots,F_k)} O_W\ ]$$

and define a closed subscheme Y to be $(|Supp(O_W/I_{Y/W})|, O_W/I_{Y/W})$. If $Y \subseteq X$, then the scheme W is a PG-shell of the scheme X.

Proof. Just using Lemma 1.3, Lemma 1.12, and the fact that there are natural ring homomorphisms $R_W \to R_X \to R_Y = R_W/\Sigma F_i R_W$. Here we need a bit of effort to show that the sequence $\{F_1, \dots, F_k\}$ is also a R_W-regular sequence. ∎

§2 Several Results.

In this section, let us consider PGS-extensions more precisely.

Lemma 2.1 *Take a short exact sequece of finite graded S-modules :*

(#-13) $$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0.$$

Then, the following two conditions are equivalent.

(2.1.1) *After tensoring S/S_+ to the sequence (#-13), we still have an exact sequence :*

(#-14) $$0 \longrightarrow M_1 \otimes (S/S_+) \xrightarrow{\overline{\varphi}} M_2 \otimes (S/S_+) \xrightarrow{\overline{\psi}} M_3 \otimes (S/S_+) \longrightarrow 0.$$

(2.1.2) *A short exact sequence :*

(#-15) $$0 \longrightarrow M_1/(S_+ \cdot M_1) \xrightarrow{\widehat{\varphi}} M_2/(S_+ \cdot M_1) \xrightarrow{\widehat{\psi}} M_3 \longrightarrow 0$$

splits after taking quotients of the modules in the sequence (#-13) by the submodule $(S_+ \cdot M_1)$.

Proof. First we assume the condition (2.1.2). Then, tensoring (S/S_+) to the splitting short exact sequence (#-15), we obtain the short exact sequence (#-14).

Now we show the converse. Let us compare the sequence (#-15) with the sequence (#-14) in the exact commutative diagram (#-16) below.

(#-16) $$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1/(S_+ \cdot M_1) & \xrightarrow{\widehat{\varphi}} & M_2/(S_+ \cdot M_1) & \xrightarrow{\widehat{\psi}} & M_3 & \longrightarrow & 0 \\ & & \Vert & & f_2 \downarrow & & \downarrow f_3 & & \\ 0 & \longrightarrow & M_1 \otimes (S/S_+) & \xrightarrow{\overline{\varphi}} & M_2 \otimes (S/S_+) & \xrightarrow{\overline{\psi}} & M_3 \otimes (S/S_+) & \longrightarrow & 0 \end{array}$$

Since the exact sequence (#-14) is that of the finite dimensional vector spaces on the field S/S_+, we have a S/S_+-linear splitting homomorphism $\overline{\sigma} : M_2 \otimes (S/S_+) \to M_1 \otimes (S/S_+)$, namely $\overline{\sigma} \circ \overline{\varphi} = 1_{\overline{M_1}}$, where the homomorphism $1_{\overline{M_1}}$ is the identity map of the module $\overline{M_1} := M_1 \otimes (S/S_+)$. Now we put $\widehat{\sigma} := \overline{\sigma} \circ f_2 : M_2/(S_+ \cdot M_1) \to M_1 \otimes (S/S_+)$. Then $\widehat{\sigma} \circ \widehat{\varphi} = \overline{\sigma} \circ f_2 \circ \widehat{\varphi} = \overline{\sigma} \circ \overline{\varphi} = 1_{\overline{M_1}}$, which means that the homomorphism $\widehat{\sigma}$ is a splitting homomorphism of the sequence (#-15). ∎

Lemma 2.2 *Let us consider the following exact commutative diagram of graded S-modules :*

(#-17)
$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M' \arrow[r, "\varphi"] \arrow[u] & M \arrow[r, "\psi"] \arrow[u] & M'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F' \arrow[r, "\eta"] \arrow[u, "\mu'"] & F \arrow[r] \arrow[u, "\mu"] & F'' \arrow[r] \arrow[u, "\mu''"] & 0 \\
0 \arrow[r] & K' \arrow[r, "\kappa"'] \arrow[u, "\nu'"] & K \arrow[r, "\lambda"'] \arrow[u, "\nu"] & K'' \arrow[r] \arrow[u, "\nu''"] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Assume that the modules F', F and F'' are S-free, and the sequence $0 \to M' \to M \to M'' \to 0$ is a PGS-extension. Then, the sequence $0 \to K' \to K \to K'' \to 0$ is also a PGS-extension.

Proof. Set $q \geq 1$ and apply $Tor_*^S(-, S/S_+)$ to the diagram (#-17) and get an exact commutative diagram:

(#-18)
$$\begin{tikzcd}
 & Tor_q^S(F', S/S_+) = 0 \arrow[r] & Tor_q^S(F, S/S_+) = 0 \arrow[r] & Tor_q^S(F'', S/S_+) = 0 & \\
 & Tor_q^S(K', S/S_+) \arrow[r] \arrow[u] & Tor_q^S(K, S/S_+) \arrow[r] \arrow[u] & Tor_q^S(K'', S/S_+) \arrow[u] & \\
0 \arrow[r] & Tor_{q+1}^S(M', S/S_+) \arrow[r] \arrow[u, "\cong"] & Tor_{q+1}^S(M, S/S_+) \arrow[r] \arrow[u, "\cong"] & Tor_{q+1}^S(M'', S/S_+) \arrow[r] \arrow[u, "\cong"] & 0 \\
 & Tor_{q+1}^S(F', S/S_+) = 0 \arrow[r] \arrow[u] & Tor_{q+1}^S(F, S/S_+) = 0 \arrow[r] \arrow[u] & Tor_{q+1}^S(F'', S/S_+) = 0. \arrow[u] &
\end{tikzcd}$$

Thus we get a short exact sequence $0 \to Tor_q^S(K', S/S_+) \to Tor_q^S(K, S/S_+) \to Tor_q^S(K'', S/S_+) \to 0$ for $q \geq 1$. In particular, we have the surjectivity of $Tor_1^S(K, S/S_+) \to Tor_1^S(K'', S/S_+)$, which and the long Tor exact sequence induced by tensoring S/S_+ to the sequence $0 \to K' \to K \to K'' \to 0$ imply the exactness of $0 \to Tor_0^S(K', S/S_+) \to Tor_0^S(K, S/S_+) \to Tor_0^S(K'', S/S_+) \to 0$. ∎

Let us consider the converse implication of Lemma 2.2.

Theorem 2.3 *Recall the diagram (#-17). Assume that the modules* F', F *and* F'' *are* S*-free, and the sequence* $0 \to K' \to K \to K'' \to 0$ *is a PGS-extension. Moreover, suppose* $\nu(K) \subseteq (S_+ \cdot F)$*, or equivalently the homomorphism induces an isomorphism* $\overline{\mu} : F \otimes (S/S_+) \xrightarrow{\sim} M \otimes (S/S_+)$*. Then the sequence* $0 \to M' \to M \to M'' \to 0$ *is also a PGS-extension.*

Proof. First we set $q \geq 2$ and apply $Tor_*^S(-, S/S_+)$ to the diagram (#-17) and get an exact commutative diagram:

(#-19)

$$\begin{tikzcd}
 & Tor_{q-1}^S(F', S/S_+) = 0 \arrow[r] & Tor_{q-1}^S(F, S/S_+) = 0 \arrow[r] & Tor_{q-1}^S(F'', S/S_+) = 0 & \\
0 \arrow[r] & Tor_{q-1}^S(K', S/S_+) \arrow[r] \arrow[u] & Tor_{q-1}^S(K, S/S_+) \arrow[r] \arrow[u] & Tor_{q-1}^S(K'', S/S_+) \arrow[r] \arrow[u] & 0 \\
 & Tor_q^S(M', S/S_+) \arrow[r] \arrow[u, "\cong"] & Tor_q^S(M, S/S_+) \arrow[r] \arrow[u, "\cong"] & Tor_q^S(M'', S/S_+) \arrow[u, "\cong"] & \\
 & Tor_q^S(F', S/S_+) = 0 \arrow[r] \arrow[u] & Tor_q^S(F, S/S_+) = 0 \arrow[r] \arrow[u] & Tor_q^S(F'', S/S_+) = 0, \arrow[u] &
\end{tikzcd}$$

which brings us the short exact sequence $0 \to Tor_q^S(M', S/S_+) \to Tor_q^S(M, S/S_+) \to Tor_q^S(M'', S/S_+) \to 0$ for $q \geq 2$. The remaining cases are $q = 0$ and $q = 1$. From the long Tor exact sequence induced by tensoring S/S_+ to the sequence $0 \to M' \to M \to M'' \to 0$, it is enough to show the injectivity of the induced homomorphism $\overline{\varphi} : M' \otimes S/S_+ \to M \otimes S/S_+$. Now we tensor S/S_+ to the diagram (#-17), use the assumption $\nu(K) \subseteq (S_+ \cdot F)$, and get an exact commutative diagram:

(#-20)

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & M' \otimes (S/S_+) \arrow[r, "\overline{\varphi}"] \arrow[u] & M \otimes (S/S_+) \arrow[r, "\overline{\psi}"] \arrow[u] & M'' \otimes (S/S_+) \arrow[r] \arrow[u] & 0 \\
0 = Tor_1^S(F'', S/S_+) \arrow[r] & F' \otimes (S/S_+) \arrow[r, "\overline{\eta}"] \arrow[u, "\overline{\mu'}"] & F \otimes (S/S_+) \arrow[r] \arrow[u, "\overline{\mu}", "\cong"'] & F'' \otimes (S/S_+) \arrow[r] \arrow[u, "\overline{\mu''}"] & 0 \\
0 \arrow[r] & K' \otimes (S/S_+) \arrow[r, "\overline{\kappa}"'] \arrow[u, "\overline{\nu'}"] & K \otimes (S/S_+) \arrow[r, "\overline{\lambda}"'] \arrow[u, "0=\overline{\nu}"] & K'' \otimes (S/S_+) \arrow[r] \arrow[u, "\overline{\nu''}"] & 0.
\end{tikzcd}$$

Since $\overline{\nu} = 0$, we have $\overline{\eta} \circ \overline{\nu'} = 0$. Using the injectivity of $\overline{\eta}$, we see $\overline{\nu'} = 0$, or equivalently the homomorphism $\overline{\mu'}$ is an isomorphism. Now the injectivity of $\overline{\varphi}$ is obvious from the injectivity of $\overline{\eta}$. ∎

Corollary 2.4 *In the exact commutative diagram (#-17), we assume that the modules* F', F *and* F'' *are* S*-free, and* $\nu(K) \subseteq (S_+ \cdot F)$*. Then the sequence* $0 \to M' \to M \to M'' \to 0$ *is a PGS-extension if and only if the sequence* $0 \to K' \to K \to K'' \to 0$ *is a PGS-extension.*

Remark 2.5 *In Theorem 2.3, the assumption $\nu(K) \subseteq (S_+ \cdot F)$ is crucial. Without this assumption, we can construct a counter-example, which will be given in Example 3.2.*

Let us give a criterion for PGS-extensions in terms of minimal graded S-free resolutions.

Theorem 2.6 *Take a short exact sequence*

$$(\#\text{-}21) \qquad 0 \longrightarrow M' \xrightarrow{\ \varphi\ } M \xrightarrow{\ \psi\ } M'' \longrightarrow 0$$

of finite graded S-modules. Then, the following three conditions are equivalent.

(2.6.1) *The sequence (#-21) is a PGS-extension.*

(2.6.2) *First we set $K_0' := M'$, $K_0 := M$, $K_0'' := M''$ $\varphi_0 := \varphi$ and $\psi_0 := \psi$. Define inductively a short exact sequence $0 \to K_k' \stackrel{\varphi_k}{\to} K_k \stackrel{\psi_k}{\to} K_k'' \to 0$ for any $k \geq 0$ as follows. Take* any *short exact sequence $0 \to F_k' \stackrel{\Phi_k}{\to} F_k \stackrel{\Psi_k}{\to} F_k'' \to 0$ of graded S-free modules F_k', F_k and F_k'' which forms an exact commutative diagram*

$$(\#\text{-}22) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K_k' & \xrightarrow{\varphi_k} & K_k & \xrightarrow{\psi_k} & K_k'' & \longrightarrow & 0 \\ & & \mu_k' \uparrow & & \mu_k \uparrow & & {\mu_k'}' \uparrow & & \\ 0 & \longrightarrow & F_k' & \xrightarrow{\Phi_k} & F_k & \xrightarrow{\Psi_k} & F_k'' & \longrightarrow & 0 \end{array}$$

where the homomorphisms $\mu_k' : F_k' \to K_k'$, $\mu_k : F_k \to K_k$, and $\mu_k'' : F_k'' \to K_k''$ are surjective. Now we set $K_{k+1}' := Ker(\mu_k')$, $K_{k+1} := Ker(\mu_k)$, $K_{k+1}'' := Ker(\mu_k'')$, $\varphi_{k+1} := \Phi_k|_{K_{k+1}'}$ and $\psi_{k+1} := \Psi_k|_{K_{k+1}}$, which induces a short exact sequence $0 \to K_{k+1}' \stackrel{\varphi_{k+1}}{\to} K_{k+1} \stackrel{\psi_{k+1}}{\to} K_{k+1}'' \to 0$ by Snake Lemma. Then, we always have a short exact sequence $0 \to K_k' \otimes (S/S_+) \stackrel{\overline{\varphi_k}}{\to} K_k \otimes (S/S_+) \stackrel{\overline{\psi_k}}{\to} K_k'' \otimes (S/S_+) \to 0$ for any $k \geq 0$, or equivalently, the sequence $0 \to K_k'/(S_+ \cdot K_k') \stackrel{\widehat{\varphi_k}}{\to} K_k/(S_+ \cdot K_k') \stackrel{\widehat{\psi_k}}{\to} K_k'' \to 0$ always splits for any $k \geq 0$ (cf. Lemma 2.1).

(2.6.3) *There exist minimal graded S-free resolutions $\mathbb{F}'_\bullet \to M'$, $\mathbb{F}_\bullet \to M$, $\mathbb{F}''_\bullet \to M''$, and complex homomorphisms $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ and $\Psi_\bullet : \mathbb{F}_\bullet \to \mathbb{F}''_\bullet$ induced by φ and ψ which satisfy that the sequence $0 \to F_k' \stackrel{\Phi_k}{\to} F_k \stackrel{\Psi_k}{\to} F_k'' \to 0$ is exact for any $k \geq 0$.*

Proof. First we show the implication (2.6.3) $\Rightarrow$ (2.6.1). Since all of the differential maps μ_k', μ_k, μ_k'' of the minimal graded S-free resolutions $\mathbb{F}'_\bullet = \{(F_k', \mu_k')\}_{k\geq 0}$, $\mathbb{F}_\bullet = \{(F_k, \mu_k)\}_{k \geq 0}$, and $\mathbb{F}''_\bullet = \{(F_k'', \mu_k'')\}_{k\geq 0}$, are killed by tensoring S/S_+, for any $q \geq 0$, we obtain an exact commutative diagram

$$(\#\text{-}23) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & F_q' \otimes (S/S_+) & \xrightarrow{\overline{\Phi_q}} & F_q \otimes (S/S_+) & \xrightarrow{\overline{\Psi_q}} & F_q'' \otimes (S/S_+) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & Tor_q^S(M', S/S_+) & \xrightarrow[\varphi_*]{} & Tor_q^S(M, S/S_+) & \xrightarrow[\psi_*]{} & Tor_q^S(M'', S/S_+), & & \end{array}$$

which shows that the sequence (#-21) is a PGS-extension.

Next we see the implication (2.6.1) $\Rightarrow$ (2.6.2). It is enough to apply Lemma 2.2 inductively on k and see that the sequence . $0 \to K'_k \overset{\varphi_k}{\to} K_k \overset{\psi_k}{\to} K''_k \to 0$ a PGS-extension.

The remain is to show the implication (2.6.2) $\Rightarrow$ (2.6.3). We will construct inductively minimal graded S-free resolutions $\mathbb{F}'_\bullet \to M'$, $\mathbb{F}_\bullet \to M$, $\mathbb{F}''_\bullet \to M''$, and complex homomorphisms $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ and $\Psi_\bullet : \mathbb{F}_\bullet \to \mathbb{F}''_\bullet$ simultaneously. For the modules $K'_0 = M'$ and $K_0 = M$, take graded S-free modules F'_0, F_0, surjective graded S-linear homomorphisms $\mu'_0 : F'_0 \to M'$, $\mu_0 : F_0 \to M$ which induce isomorphisms $\overline{\mu'_0} : F'_0 \otimes (S/S_+) \tilde{\to} M' \otimes (S/S_+)$, $\overline{\mu_0} : F_0 \otimes (S/S_+) \tilde{\to} M \otimes (S/S_+)$ and an S-linear lift $\Phi_0 : F'_0 \to F_0$ of the homomorphism φ. By the argument in the proof of Lemma 1.7, we see that the quotient graded S-module $F''_0 := F_0/\Phi_0(F'_0)$ is S-free and the module F'_0 is a direct summand of F_0. Set $\Psi_0 : F_0 \to F''_0$ to be the canonical quotient homomorphism. Then we obtain naturally a graded S-linear homomorphism $\mu''_0 : F''_0 \to K''_0 = M''$ which forms an exact commutative diagram

$$
\text{(\#-24)} \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \\
 & & \mu'_0 \uparrow & & \mu_0 \uparrow & & \mu''_0 \uparrow & & \\
0 & \longrightarrow & F'_0 & \xrightarrow{\Phi_0} & F_0 & \xrightarrow{\Psi_0} & F''_0 & \longrightarrow & 0.
\end{array}
$$

Since the induced homomorphisms $\overline{\mu'_0}$ and $\overline{\mu_0}$ are isomorphic, the homomorphism $\overline{\mu''_0}$ is also isomorphic by using the diagram (#-24) above after tensored with S/S_+. Then it is easy to see that the homomorphism μ''_0 is surjective, $K'_1 \subseteq S_+ \cdot F'_0$, $K_1 \subseteq S_+ \cdot F_0$ and $K''_1 \subseteq S_+ \cdot F''_0$. Use the assumption (2.6.2), replace M', M, M'' with K'_1, K_1, K''_1, respectively and apply the same argument. Then we obtain an exact commutative diagram

$$
\text{(\#-25)} \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & K'_1 & \xrightarrow{\varphi_1} & K_1 & \xrightarrow{\psi_1} & K''_1 & \longrightarrow & 0 \\
 & & \mu'_1 \uparrow & & \mu_1 \uparrow & & \mu''_1 \uparrow & & \\
0 & \longrightarrow & F'_1 & \xrightarrow{\Phi_1} & F_1 & \xrightarrow{\Psi_1} & F''_1 & \longrightarrow & 0
\end{array}
$$

with the property that the induced homomorphisms $\overline{\mu'_1}$, $\overline{\mu_1}$ and $\overline{\mu''_1}$ are isomorphic. We continue this inductive argument with replacing K'_k, K_k, K''_k by K'_{k+1}, K_{k+1}, K''_{k+1}, respectively, and obtain minimal graded S-free resolutions $\mathbb{F}'_\bullet \to M'$, $\mathbb{F}_\bullet \to M$, $\mathbb{F}''_\bullet \to M''$, and complex homomorphisms $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ and $\Psi_\bullet : \mathbb{F}_\bullet \to \mathbb{F}''_\bullet$ with the desired properties. ∎

Remark 2.7 *In classical text books on homological algebra* (cf. e.g. [1], [2], [6], [7])*, for a given short exact sequence of (S-)modules :* $0 \to M' \overset{\varphi}{\to} M \overset{\psi}{\to} M'' \to 0$*, we often construct simultaneous projective resolutions, namely projective resolutions* $\mathbb{F}'_\bullet \to M'$*,* $\mathbb{F}_\bullet \to M$*,* $\mathbb{F}''_\bullet \to M''$*, and complex homomorphisms* $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ *and* $\Psi_\bullet : \mathbb{F}_\bullet \to \mathbb{F}''_\bullet$ *which are compatible with the homomorphisms* φ *and* ψ *and the sequence* $0 \to F'_k \overset{\Phi_k}{\to} F_k \overset{\Psi_k}{\to} F''_k \to 0$ *is exact for any* $k \geq 0$*. Following to this standard construction, we can make the S-free resolutions* $\mathbb{F}'_\bullet \to M'$ *and* $\mathbb{F}''_\bullet \to M''$ *minimal. However, as we saw in Theorem 2.6, we can not always make the middle part* $\mathbb{F}_\bullet \to M$ *minimal.*

Definition 2.8 *For a PGS-extension* $0 \to M' \to M \to M'' \to 0$*, by Theorem 2.6, we obtain minimal graded S-free resolutions* $\mathbb{F}'_\bullet \to M'$*,* $\mathbb{F}_\bullet \to M$*,* $\mathbb{F}''_\bullet \to M''$*, and complex homomorphisms* $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ *and* $\Psi_\bullet : \mathbb{F}_\bullet \to \mathbb{F}''_\bullet$ *as in the condition* (2.6.3)*. The combination of these three minimal graded S-free resolutions and two complex homomorphisms* $0 \to \mathbb{F}'_\bullet \overset{\Phi_\bullet}{\to} \mathbb{F}_\bullet \overset{\Psi_\bullet}{\to} \mathbb{F}''_\bullet \to 0$ *is called simply a* <u>simultaneous minimal</u> graded S-free resolutions *of the PGS-extension.*

The following result is obvious but shows the importance of monoPGS-homomorphisms and epiPGS-homomorphisms.

Lemma 2.9 *Let us take finite graded S-modules L, M, and a graded S-linear homomorphism $\varphi : L \to M$. Then, we have the following facts.*

(2.9.1) *If the homomorphism φ is a monoPGS-homomorphism, then there are an inequality of the homological dimensions : $hd_S(L) \leq hd_S(M)$, an inequality of depth : $depth_S(L) \geq depth_S(M)$, and an inequality of Castelnuovo-Mumford regularity : $reg^{CM}(L) \leq reg^{CM}(M)$. In this case, if the module M is S-free, then the module L is also S-free and is a direct summand of M.*

(2.9.2) *If the homomorphism φ is a epiPGS-homomorphism, then there are an inequality of the homological dimensions : $hd_S(L) \geq hd_S(M)$, an inequality of depth : $depth_S(L) \leq depth_S(M)$, and an inequality of Castelnuovo-Mumford regularity : $reg^{CM}(L) \geq reg^{CM}(M)$. In this case, if the module L is S-free, then the module M is also S-free.*

Proof. Take minimal graded S-free resolutions $\mathbb{F}^L_\bullet \to L$ and $\mathbb{F}^M_\bullet \to M$ of the modules L and M, respectively. If the homomorphism φ is a monoPGS-homomorphism, then the complex $\mathbb{F}^L_\bullet$ is a subcomplex of $\mathbb{F}^M_\bullet$ by Lemma 1.7 and Theorem 2.6. Also if the homomorphism φ is an epiPGS-homomorphism, then the complex $\mathbb{F}^M_\bullet$ is a quotient complex $\mathbb{F}^L_\bullet$ by Lemma 1.5 and Theorem 2.6. Thus the inequality of the homological dimensions is obvious. On the inequality of depth, apply the Auslander-Buchsbaum formula (cf. [8]). To get the inequality of Castelnuovo-Mumford regularity, use Eisenbud-Goto criterion (cf. [3]). ∎

Lemma 2.10 *Let M be a finite graded S-modules, $L = \oplus Se_i$ a graded S-free module of finite rank, and $\varphi : L \to M$, $\psi : M \to L$ S-linear homomorphisms. Then, the homomorphism φ is a monoPGS-homomorphism if and only if the induced homomorphism $\overline{\varphi} : L \otimes (S/S_+) \to M \otimes (S/S_+)$ is injective, namely the set $\{\varphi(e_i)\}_i$ forms a part of minimal generators of M. The homomorphism ψ is an epiPGS-homomorphism if and only if $\overline{\psi} : M \otimes (S/S_+) \to L \otimes (S/S_+)$ is surjective, namely the homorphism ψ is surjective, or equivalently the module has a direct summand which is isomorphic to L via the homomorphism ψ.*

Proof. Obvious. ∎

Theorem 2.11 *Let us consider a short exact sequence*

$$0 \longrightarrow M' \xrightarrow{\ \varphi\ } M \xrightarrow{\ \psi\ } M'' \longrightarrow 0 \tag{\#-26}$$

of finite graded S-modules. Then, the sequence (#-26) is a PGS-extension if and only if the induced sequence

$$0 \longrightarrow Ext^q_S(M'', S/S_+) \xrightarrow{\ \psi^*\ } Ext^q_S(M, S/S_+) \xrightarrow{\ \varphi^*\ } Ext^q_S(M', S/S_+) \longrightarrow 0 \tag{\#-27}$$

is exact for any $q \geq 0$.

Proof. First we assume that the sequence (#-26) is a PGS-extension. Then, by Theorem 2.6, we obtain a simultaneous minimal graded S-free resolutions $0 \to \mathbb{F}'_\bullet \overset{\Phi_\bullet}{\to} \mathbb{F}_\bullet \overset{\Psi_\bullet}{\to} \mathbb{F}''_\bullet \to 0$ of the PGS-extension. Let us denote more precisely these minimal graded S-free resolutions by $\mathbb{F}'_\bullet = \{(F'_k, \mu'_k)\}_{k \geq 0}$,

$\mathbb{F}_\bullet = \{(F_k, \mu_k)\}_{k\geq 0}$, and $\mathbb{F}''_\bullet = \{(F''_k, \mu''_k)\}_{k\geq 0}$. To get the long Ext sequence induced from (#-26), we apply the functor $Hom_S(-, S/S_+)$ to this simultaneous minimal graded S-free resolutions. Then all of the differential maps μ'^*_k, μ^*_k, μ''^*_k of the complexes $Hom_S(\mathbb{F}'_\bullet, S/S_+) = \{(Hom_S(F'_k, S/S_+), \mu'^*_k)\}_{k\geq 0}$, $Hom_S(\mathbb{F}_\bullet, S/S_+) = \{(Hom_S(F_k, S/S_+), \mu^*_k)\}_{k\geq 0}$, and $Hom_S(\mathbb{F}''_\bullet, S/S_+) = \{(Hom_S(F''_k, S/S_+), \mu''^*_k)\}_{k\geq 0}$, are killed. Thus we have an exact commutative diagram

(#-28)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_S(F''_q, S/S_+) & \xrightarrow{\Psi^*_q} & Hom_S(F_q, S/S_+) & \xrightarrow{\Phi^*_q} & Hom_S(F'_q, S/S_+) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & Ext^q_S(M'', S/S_+) & \xrightarrow[\psi^*]{} & Ext^q_S(M, S/S_+) & \xrightarrow[\varphi^*]{} & Ext^q_S(M', S/S_+), & & \end{array}$$

which implies the exactness of (#-27).

To show the implication of the converse direction, we assume that the exactness of (#-27) for any $q \geq 0$ holds. Tensoring S/S_+ to the sequence (#-26), we obtain

(#-29) $$\overline{M'} = M' \otimes (S/S_+) \xrightarrow{\overline{\varphi}} \overline{M} = M \otimes (S/S_+) \xrightarrow{\overline{\psi}} \overline{M''} = M'' \otimes (S/S_+) \longrightarrow 0.$$

Let us see that the homomorphism $\overline{\varphi}$ is injective. Recall the exactness (#-27) in the case $q = 0$:

(#-30)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_S(M'', S/S_+) & \xrightarrow{\psi^*} & Hom_S(M, S/S_+) & \xrightarrow{\varphi^*} & Hom_S(M', S/S_+) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & Hom_S(\overline{M''}, S/S_+) & \xrightarrow{\overline{\psi}^\vee} & Hom_S(\overline{M}, S/S_+) & \xrightarrow{\overline{\varphi}^\vee} & Hom_S(\overline{M'}, S/S_+), & & \end{array}$$

where the symbols $\overline{\varphi}^\vee$ and $\overline{\psi}^\vee$ denotes the homomorphisms induced from the homomorphisms $\overline{\varphi}$ and $\overline{\psi}$ in the diagram (#-29). The diagram (#-30) shows the surjectivity of the homomorphism $\overline{\varphi}^\vee$. Since the modules $M' \otimes (S/S_+)$ and $M \otimes (S/S_+)$ are the finite dimensional vector spaces over the field S/S_+, taking duals over the field S/S_+, we obtain the injectivity of the homomorphism $\overline{\varphi} = \overline{\varphi}^{\vee\vee}$.

In case that the module M' is S-free, namely the homological dimension $hd_S(M')$ is zero, by Lemma 2.10, the injectivity of the homomorphism $\overline{\varphi}$ shows that the injective homomorphism $\varphi : M' \to M$ is a monoPGS-homomorphism, which implies the sequence (#-26) is a PGS-extension.

Now we will proceed by induction on $h' := hd_S(M')$ and assume that $h' \geq 1$ and our Theorem holds if $hd_S(M') \leq h' - 1$. Since we have already proven that the sequence $0 \to M' \otimes (S/S_+) \xrightarrow{\overline{\varphi}} M \otimes (S/S_+) \xrightarrow{\overline{\psi}} M'' \otimes (S/S_+) \to 0$ is exact, applying the similar argument for the implication (2.6.2) $\Rightarrow$ (2.6.3) in the proof of Theorem 2.6, we obtain an exact commutative diagram

(#-31)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \\
 & & \mu_0' \uparrow & & \mu_0 \uparrow & & \mu_0'' \uparrow & & \\
0 & \longrightarrow & F_0' & \xrightarrow{\Phi_0} & F_0 & \xrightarrow{\Psi_0} & F_0'' & \longrightarrow & 0 \\
 & & \nu_0' \uparrow & & \nu_0 \uparrow & & \nu_0'' \uparrow & & \\
0 & \longrightarrow & K_1' & \xrightarrow{\varphi_1} & K_1 & \xrightarrow{\psi_1} & K_1'' & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the modules F_0', F_0 and F_0'' are S-free and the induced homomorphisms $\overline{\mu_0'}$, $\overline{\mu_0}$ and $\overline{\mu_0''}$ from the homomorphisms μ_0', μ_0 and μ_0'' after tensoring S/S_+ to the diagram (#-31) are isomorphic, the symbols ν_0', ν_0, and ν_0'' denote inclusion homomorphisms. Applying the functor $Hom_S(-, S/S_+)$ to the diagram (#-31) and using the similar argument in the proof of Lemma 2.2 with replacing $Tor_*^S(-, S/S_+)$ by $Ext_S^*(-, S/S_+)$, we see that for any $q \geq 0$,

(#-32)
$$\begin{array}{ccccccccc}
 & & Ext_S^q(K_1'', S/S_+) & \xrightarrow{\psi_1^*} & Ext_S^q(K_1, S/S_+) & \xrightarrow{\varphi_1^*} & Ext_S^q(K_1', S/S_+) & & \\
 & & \cong \downarrow \delta' & & \cong \downarrow \delta & & \cong \downarrow \delta'' & & \\
0 & \longrightarrow & Ext_S^{q+1}(M'', S/S_+) & \xrightarrow{\psi^*} & Ext_S^{q+1}(M, S/S_+) & \xrightarrow{\varphi^*} & Ext_S^{q+1}(M', S/S_+) & \longrightarrow & 0,
\end{array}$$

which implies that the sequence $0 \to K_1' \overset{\varphi_1}{\to} K_1 \overset{\psi_1}{\to} K_1'' \to 0$ has the property of the exactness (#-27) for any $q \geq 0$. Since $hd_S(K_1') = hd_S(M') = h' - 1$, our induction hypothesis tells that the sequence $0 \to K_1' \to K_1 \to K_1'' \to 0$ is a PGS-extension. Then we apply Theorem 2.3 and see that the sequence (#-26) is also a PGS-extension. ∎

Corollary 2.12 *Recall the short exact sequence (#-26) of finite graded S-modules. Take the extension class $\varepsilon \in Ext_S^1(M'', M')$ of this sequence. Then, the sequence (#-26) is a PGS-extension if and only if for any class $\gamma \in Ext_S^q(M', S/S_+)$, the element $\gamma \circ \varepsilon \in Ext_S^{q+1}(M'', S/S_+)$ is zero.*

§3 Examples.

In this section, we give two counter-examples against naive expectations on monoPGS-homomorphisms and on PGS-extensions.

The highest non-zero Tor-group (or Ext-group) often plays a dominant role in homological phenomena of commutative ring theory. For example, take a noetherian local ring $(A, \mathfrak{m}, k)$ and a finite graded A-module M, then the equality $hd_A(M) = max\{\ q\ |Tor_q^A(M, k) \neq 0\}$ holds. Thus one might have a naive expectation as follows.

Working Problem 3.1 *Take a finite graded S-module M and its graded S-submodule M'. Set $r := hd_S(M')$, and $\varphi : M' \hookrightarrow M$ to be the inclusion homomorphism, and assume that the induced homomorphism $\varphi_* : Tor_r^S(M', S/S_+) \to Tor_r^S(M, S/S_+)$ is injective. Then is the module M' a PGS-submodule of M ? (In another words, is the sequence $0 \to M' \to M \to M/M' \to 0$ a PGS-extension ?)*

Now we give a counter example to this naive expectation, namely Working Problem 3.1. This example gives also a counter-example desired in Remark 2.5

Example 3.2 *Let us consider the twisted cubic curve $X := \mathbb{P}^1(\mathbb{C}) \ni [s : t] \longmapsto [x : y : z : w] = [s^3 : s^2t : st^2 : t^3] \in \mathbb{P}^3(\mathbb{C}) = Proj(S) = P$ where $S = \mathbb{C}[x, y, z, w]$ and a closed subscheme $Y := \{[1 : 0 : 0 : 0], [0 : 0 : 0 : 1]\}$ which is two points on X. Set $M' := \mathbb{I}_X$ and $M := \mathbb{I}_Y$, namely $M' = (f_1, f_2, f_3)S$ and $M = (g_1, g_2, g_3)S$ where $f_1 := xz - y^2$, $f_2 := xw - yz$, $f_3 := yw - z^2$, $g_1 := y$, $g_2 := z$, $g_3 := xw$. Then, it is well-known that the module M' has the minimal graded S-free resolution $\mathbb{F}'_\bullet \to M'$:*

$$0 \longrightarrow F'_1 = \oplus_{j=1}^2 S[\tau_j] \xrightarrow{\mu'_1} F'_0 = \oplus_{i=1}^3 S[f_i] \xrightarrow{\mu'_0} M' \longrightarrow 0, \tag{\#-33}$$

where $\tau_1 = w[f_1] - z[f_2] + y[f_3] \in F'_0$ and $\tau_2 = z[f_1] - y[f_2] + x[f_3]$. It shows us that $hd_S(M') = 1$.

Since the scheme Y is a complete intersection, the module M has the Koszul resolution as its minimal graded S-free resolution $\widetilde{\mathbb{F}_\bullet} \to M$:

$$\begin{array}{ccccccc} 0 \longrightarrow \widetilde{F_2} = S[g_1] \wedge [g_2] \wedge [g_3] & \longrightarrow & \widetilde{F_1} = \bigoplus_{1 \le i < j \le 3} S[g_i] \wedge [g_j] & \longrightarrow & \widetilde{F_0} = \bigoplus_{i=1}^{3} S[g_i] \\ & \longrightarrow & M & \longrightarrow & 0. \end{array} \tag{\#-34}$$

However, for later use, by adding an acyclic S-free comlex, we expand this resolution to a non-minimal free resolution $\mathbb{F}_\bullet \to M$:

$$0 \longrightarrow F_2 \xrightarrow{\mu_2} F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} M \longrightarrow 0, \tag{\#-35}$$

where $F_0 := \widetilde{F_0} \oplus \bigoplus_{j=1}^{3} S[f_j]$, $\quad F_1 := \widetilde{F_1} \oplus \bigoplus_{k=1}^{3} S[h_j]$, $\quad F_2 := \widetilde{F_2}$, and $h_1 = [f_1] - x[g_2] + y[g_1]$, $h_2 = [f_2] - [g_3] + z[g_1]$, $h_3 = [f_3] - w[g_1] + z[g_2]$. Then we construct a complex homomorphism $\Phi_\bullet : \mathbb{F}'_\bullet \to \mathbb{F}_\bullet$ induced from the inclusion homomorphism $\varphi : M' \to M$. The homomorphism $\Phi_0 : F'_0 \to F_0$ is defined naturally, namely $\Phi_0([f_i]) = [f_i]$. The homomorphism $\Phi_1 : F'_1 \to F_1$ is defined as follows.

$$\begin{array}{rcl} \Phi_1([\tau_1]) & = & -[g_2] \wedge [g_3] - z[g_1] \wedge [g_2] + w[h_1] - z[h_2] + y[h_3] \\ \Phi_1([\tau_2]) & = & -[g_1] \wedge [g_3] + z[h_1] - y[h_2] + x[h_3] \end{array} \tag{\#-36}$$

Using $Tor_1^S(M', S/S_+) = H_1(\mathbb{F}'_\bullet \otimes (S/S_+))$ and $Tor_1^S(M, S/S_+) = H_1(\mathbb{F}_\bullet \otimes (S/S_+))$, we see that

$$\begin{array}{rcl} Tor_1^S(M', S/S_+) & \cong & \bigoplus_{i=1}^{2} (S/S_+)[\tau_i] \\ Tor_1^S(M, S/S_+) & \cong & \bigoplus_{1 \le j < k \le 3} (S/S_+)[g_j] \wedge [g_k]. \end{array} \tag{\#-37}$$

Thus, from the formula (#-36)*, the induced homomorphism* $\varphi_* : Tor_1^S(M', S/S_+) \to Tor_1^S(M, S/S_+)$ *is described as* $\varphi_*([\tau_1]) = -[g_2] \wedge [g_3]$ *and* $\varphi_*([\tau_2]) = -[g_1] \wedge [g_3]$*, which implies the injectivity of* $\varphi_* : Tor_1^S(M', S/S_+) \to Tor_1^S(M, S/S_+)$*.*

On the other hand, the minimal generators $\{f_1, f_2, f_3\}$ *of the module* M' *does not form a part of minimal generators of* M*, which means that* $\varphi_* : Tor_0^S(M', S/S_+) \to Tor_0^S(M, S/S_+)$ *is not injective, namely the sequence* $0 \to M' \to M \to M/M' \to 0$ *is not a PGS-extension.*

Next, using this example, we give also a counter-example for the comment in Remark 2.5. Put S*-modules* $K' := Ker(\mu_0')$*,* $K := Ker(\mu_0)$*, and an injective* S*-linear homomorphism* $\kappa := \Phi_0|_{K'} : K' \to K$*. Then we have a short exact sequence* $0 \to K' \xrightarrow{\kappa} K \to K/K' \to 0$*. From the* S*-free resolutions* (#-33) *and* (#-35)*, we have*

(#-38)
$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & F_1' & \xrightarrow[\cong]{\mu_1'} & K' & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \kappa} & & \\ 0 & \longrightarrow & F_2 & \xrightarrow{\mu_2} & F_1 & \xrightarrow{\mu_1} & K & \longrightarrow & 0. \end{array}$$

After tensoring S/S_+ *to the diagram* (#-38)*, the homomorphism* $\overline{\mu_2}$ *is zero, and therefore*

(#-39)
$$\begin{array}{ccc} \oplus_{j=1}^2 (S/S_+)[\tau_j] = F_1' \otimes (S/S_+) & \xrightarrow[\cong]{\overline{\mu_1'}} & K' \otimes (S/S_+) \\ \Big\downarrow{\scriptstyle \overline{\Phi_1}} & & \Big\downarrow{\scriptstyle \overline{\kappa}} \\ \bigoplus_{1 \le i < j \le 3} (S/S_+)[g_i] \wedge [g_j] \oplus \bigoplus_{k=1}^{3} (S/S_+)[h_j] = F_1 \otimes (S/S_+) & \xrightarrow[\cong]{\overline{\mu_1}} & K \otimes (S/S_+). \end{array}$$

Since the module K' *is* S*-free by the diagram* (#-38) *and the homomorphism* $\overline{\kappa}$ *is injective by the diagram* (#-39) *and by the formula* (#-36)*, Lemma 2.10 tells us that the sequence* $0 \to K' \xrightarrow{\kappa} K \to K/K' \to 0$ *is a PGS-extension. However the sequence* $0 \to M' \to M \to M/M' \to 0$ *is not a PGS-extension as we saw above.*

Remark 3.3 *In the argument on the sequence* $0 \to K' \xrightarrow{\kappa} K \to K/K' \to 0$ *in Example 3.2 above, if we replace the non-minimal* S*-free resolution* (#-35) $\mathbb{F}_\bullet \to M$ *by the minimal* S*-free resolution* (#-34) $\widetilde{\mathbb{F}_\bullet} \to M$*, then we lose the condition that the* S*-free module* F_0' *is a direct summand of the* S*-free module* F_0 *via the homomorphism* Φ_0*.*

From Lemma 1.9, one might have a question as follows. For an S-linear homomorphism $\varphi : L \to M$ of finite graded S-modules, if the induced homomorphisms $\varphi_* : Tor_q^S(L, S/S_+) \to Tor_q^S(M, S/S_+)$ is injective for $q = 0, 1$, then is the homomorphism φ always a monoPGS-homomorphism ?

The answer is negative as we see in the next example.

Example 3.4 *Recall the twisted cubic curve* X*, its eqations* $\{f_1, f_2, f_3\}$*, and the relations of these equations* $\{\tau_1, \tau_2\}$ *in Example 3.2. Set* $M := R_X = S/\mathbb{I}_X = S/(f_1, f_2, f_3)S$ *and* $L := S/(f_1, f_2)S$*. Let us consider a natural* S*-linear homomorphism* $\varphi : L \to M$*.*

Replacing "$\to M' \to 0$*" in* (#-33) *by "*$\to S \to R_X = M \to 0$*", we obtain a minimal graded* S*-free resolution* $\mathbb{F}_\bullet^M \to M$*. Since the ring* $S/(f_1, f_2)S$ *is a complete intersection, the Koszul complex for* $\{f_1, f_2\}$ *gives a minimal graded* S*-free resolution* $\mathbb{F}_\bullet^L \to L$ *of* L*. In particular,* $F_2^M = \oplus_{i=1}^2 S[\tau_i]$ *and* $F_2^L = S[f_1] \wedge [f_2]$*. Since* $f_1[f_2] - f_2[f_1] = -x\tau_1 + y\tau_2$ *in* F_1^M*, a complex homomorphism* $\Phi_\bullet : \mathbb{F}_\bullet^L \to \mathbb{F}_\bullet^M$ *induced from the homomorphism* φ *is given by* $\Phi_0(1_S) = 1_S$*,* $\Phi_1([f_i]) = [f_i]$*, and* $\Phi_2([f_1] \wedge [f_2]) = -x[\tau_1] + y[\tau_2]$*.*

Then, for $q = 0$, $\varphi_* = \overline{\Phi_0} : S/S_+ \cong Tor_0^S(L, S/S_+) \overset{\cong}{\to} Tor_0^S(M, S/S_+) \cong S/S_+$, and for $q = 1$, $\varphi_* := \overline{\Phi_1} : Tor_1^S(L, S/S_+) \cong \oplus_{i=1}^2 (S/S_+)[f_i] \to Tor_1^S(M, S/S_+) \cong \oplus_{i=1}^3 (S/S_+)[f_i]$, which shows the injectivity of the induced homomorphism φ_* for $q = 0$ and $q = 1$.

On the other hand, for $q = 2$, the induced homomorphism $\varphi_* = \overline{\Phi_2} : Tor_2^S(L, S/S_+) \cong (S/S_+)[f_1] \wedge [f_2] \to Tor_2^S(M, S/S_+) \cong \oplus_{i=1}^2 (S/S_+)[\tau_i]$ is the zero homomorphism, and therefore not injective, which means that the homomorphism φ is not a monoPGS-homomorphism.

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