Ogus Derivations and the Explicit Syzygy Class Maps

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Abstract

Let X be a given closed subscheme with the arithmetic D_2 condition in a complex projective space $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ $(S := \mathbb{C}[Z_0, \ldots, Z_N])$. To study locally along X the sheaf of q-th syzygy module $Z^{(q)}_{X/P}$ $(q \geq 1)$ of the homogeneous coordinate ring R_X of X as an S-module, here we construct and calculate explicitly the q-syzygy class map for degree m part $\rho^{(q,m)} : \Gamma(P, Z^{(q)}_{X/P}(m)) \to H^1(X, \Omega^q_P \otimes N^\vee_{X/P}(m))$ in a local expression form with using local frames and local coordinates. This map induces an isomorphism onto the space of infinitesimal obstructions $\overline{\rho^{(q,m)}} : \Gamma(P, Z^{(q)}_{X/P}(m))/\sum_{i=0}^N Z_i \cdot \Gamma(P, Z^{(q)}_{X/P}(m-1)) \cong Tor^S_q(R_X, S/S_+)_{(m)} \to Im[\overline{\delta_{LFT}}] \subseteq H^1(X, \Omega^q_P \otimes N^\vee_{X/P}(m))$. In other words, we give good Čech representatives for the image of the $\mathbb{C}$-linear homomorphism $\rho^{(q,m)}$.

Keywords: syzygy class map, syzygy, Ogus derivations

§0 Introduction

As we presented several fundamental problems in [13], our main interest is in the "geometric structure" of a projective embedding of a given variety X. That means to study intermediate ambient schemes satisfying certain good conditions from the view point of syzygies for the embedded variety X. Such an intermediate ambient scheme is called "pregeometric shell" (abbr. PG-shell...cf. [12]).

Now let X be a closed subscheme which satisfies the arithmetic D_2 condition in a complex projective space $P = \mathbb{P}^N(\mathbb{C}) = Proj(S)$ $(S := \mathbb{C}[Z_0, \ldots, Z_N])$. From our view point, it is important to consider (locally along X) the properties of the sheaf of q-th syzygy module $Z^{(q)}_{X/P}$ $(q \geq 1)$ of the homogeneous coordinate ring R_X of X as an S-module(cf. e.g. [15]). In particular, we have to determine the minimum order of infinitesimal neighborhoods of X in P where the information of minimal generators of the S-module $\oplus_m \Gamma(P, Z^{(q)}_{X/P}(m))$ survives. This task will be attained by the joint force of this paper and the forthcoming paper. In this paper, we construct and calculate explicitly the q-syzygy class map for degree m part $\rho^{(q,m)} : \Gamma(P, Z^{(q)}_{X/P}(m)) \to H^1(X, \Omega^q_P \otimes N^\vee_{X/P}(m))$ in a local expression form with using local frames and local coordinates. This map naturally induces an isomorphism onto the space of infinitesimal obstructions $\overline{\rho^{(q,m)}} : \Gamma(P, Z^{(q)}_{X/P}(m))/\sum_{i=0}^N Z_i \cdot \Gamma(P, Z^{(q)}_{X/P}(m-1)) \cong Tor^S_q(R_X, S/S_+)_{(m)} \to Im[\overline{\delta_{LFT}}] \subseteq H^1(X, \Omega^q_P \otimes N^\vee_{X/P}(m))$ (cf. [13], [14]).

Since different homomorphisms of abelian sheaves may induce the same global map, to use an obstruction map of infinitesimal lifting for the calculation of $\rho^{(q,m)}$, all of our calculation should be carried

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out in a local expression form. Also there is another key point of our work here. That is to get a good Čech representative without ambiguity caused by the higher infinitesimal obstructions (cf. Remark 2.2). Those will be different points from the usual syzygy theory by the method of commutative rings.

It might be considered that this local calculation straightly induce a construction of a homomorphism of abelian sheaves whose global expression, after composing with the infinitesimal obstruction map, coincides with the q-syzygy class map for degree m part $\rho^{(q,m)}$. However, this is not so easy, needs much more job including a construction of new sheaves for example, and will be carried out in the forthcoming paper.

§1 Preliminaries.

At the beginning, let us summarize for convenience what will be used throughout this paper.

Notation and Conventions 1.1 *We use the terminology of [2] without mentioning so, always admit the conventions, and use the notation below for simplicity.*

(1.1.1) *Every object under consideration is defined over the field of complex numbers $\mathbb{C}$. We will work in the category of algebraic schemes and algebraically holomorphic morphisms (or rational maps) or in the categories of coherent sheaves and their ($\mathcal{O}$-linear) homomorphisms otherwise mentioned. For a sheaf M and an open set U, a local section $\tau \in \Gamma(U, M)$ is sometimes denoted by $\tau \in M$ with abbreviation if there is no need to specify the open set U.*

(1.1.2) *Let us take a complex projective scheme X of dimension n and one of its embeddings $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. The sheaf of ideals defining $j(X)$ in P and the conormal sheaf are denoted by I_X and $N^\vee_{X/P} = I_X/I_X^2$, respectively. Take a $\mathbb{C}$-basis $\{Z_0, \dots, Z_N\}$ of $H^0(P, O_P(1))$. Then we set:*

$$(\#\text{-}1)\qquad \begin{array}{rcl} S &:=& \bigoplus_{m\geq 0} H^0(P, O_P(m)) \cong \mathbb{C}[Z_0, \dots, Z_N] \\ S_+ &:=& \bigoplus_{m>0} H^0(P, O_P(m)) \cong (Z_0, \dots, Z_N)S \\ \widetilde{R_X} &:=& \bigoplus_{m\geq 0} H^0(X, O_X(m)) \\ \mathbb{I}_X &:=& \bigoplus_{m\geq 0} H^0(P, I_X(m)) \\ R_X &:=& Im[S \to \widetilde{R_X}] \cong S/\mathbb{I}_X. \end{array}$$

In the sequel, we almost always assume that the closed scheme $j(X)$ satisfies arithmetic D_2 condition, namely $R_X = \widetilde{R_X}$, otherwise mentioned.

(1.1.3) *For a graded S-module M, we denote the degree m part of M by $M_{(m)}$, namely $M = \bigoplus_{m\in\mathbb{Z}} M_{(m)}$. In particular, the S/S_+-vector space $Tor_q^S(R_X, S/S_+)_{(m)}$ which represents the minimal generators in degree m of the q-th syzygy of R_X as an S-module will be denoted by $gsyz^q_X(m)$ for simplicity. Affine sheafication (i.e. a canonically constructed $O_{Spec(S)}$-module from a S-module) of an S-module M is denoted $M^\sim$ and projective sheafication (i.e. a canonically constructed $O_{Proj(S)}$-module from a graded S-module) of a graded S-module M is denoted $M^{(\sim)}$, respectively.*

(1.1.4) *To describe the homomorphisms of free graded S-modules clearly by using matrices, we often descrbe a free graded S-module by $F = \oplus_{i=1}^n Se_i$ with using a free basis $\{e_i | deg(e_i) = m_i\}_{i=1}^n$ instead of by $F = \oplus_{i=1}^n S(-m_i)$ with using degree shifting. In this case, the isomorphism on each* *direct summand is given by* $S(-m) \supseteq S(-m)_{(k)} = S_{(k-m)} \ni g \leftrightarrow g \cdot e \in (S \cdot e)_{(k)} \subseteq S \cdot e$, *where* $deg(e) = m$.

(1.1.5) *For a coherent sheaf* E *on a projective subscheme* $V \subseteq P$, *we put:* $\Gamma_*(E) := \bigoplus_{m\in\mathbb{Z}} \Gamma(V, E(m))$, $\Gamma_{\gg 0}(E) := \bigoplus_{m\gg 0} \Gamma(V, E(m))$, $\overline{\Gamma_*(E)} = \Gamma_*(E)/(S_+ \cdot \Gamma_*(E))$, *and* $H^q_*(E) = \bigoplus_{m\in\mathbb{Z}} H^q(V, E(m))$.
Except a few cases where we have to avoid confusion, we do not use distinguished fonts comprehensively for the graded S*-modules and sheaves such as* M *and* $\mathcal{M}$, *respectively. If we need to distinguish clearly a sheaf* M *from the* S*-module* $\Gamma_*(M)$, *we use the blackboard bold font for* S*-module such as* $\mathbb{M} = \Gamma_*(M)$. *Also, for example, we denote* $M_\bullet$ *for complex of sheaves and* $\mathbb{M}_\bullet$ *for complex of S-modules, respectively.*

(1.1.6) *On an open set* $U \subseteq U_a := D_+(Z_a)$ *of the* N*-th projective space* $\mathbb{P}^N(\mathbb{C})$, *we often use the following abbreviation for summation.*

$$\sum_{j_1,j_2,\cdots,j_k}^{(a)} := \sum_{\substack{0\le j_1\le N\\ j_1\ne a}} \sum_{\substack{0\le j_2\le N\\ j_2\ne a}} \cdots \sum_{\substack{0\le j_k\le N\\ j_k\ne a}} \tag{\#-2}$$

(1.1.7) *Let* $\{M_i\}_{i=1}^k$ *be a set of abelian groups or of abelian sheaves, and* $\{f_i : M_i \to M_{i+1}\}_{i=1}^{k-1}$ *a set of their homomorphisms. Then we denote their composition as*

$$\prod_{i=1}^{k-1} f_i := f_{k-1} \circ \cdots \circ f_1 : M_1 \to M_2 \to \cdots \to M_k \tag{\#-3}$$

For the projective subscheme $X := j(X) \subseteq P = \mathbb{P}^N(\mathbb{C})$ which satisfies arithmetic D_2 condition: $R_X = \widetilde{R_X}$, take a minimal graded S-free resolution of the homogeneous coordinate ring R_X as a S-module:

$$\mathbb{F}_\bullet : \quad (R_X \xleftarrow{\varphi_0})F_0 = S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \ldots \xleftarrow{\varphi_h} F_h \longleftarrow 0, \tag{\#-4}$$

where the homomorphism φ_i denotes a graded homomorphism without grade shifting:

$$\varphi_i : F_i = \bigoplus_{j=1}^{\sigma(i)} S\, e_j^{(i)} \longrightarrow \bigoplus_{k=1}^{\sigma(i-1)} S\, e_k^{(i-1)} = F_{i-1} \tag{\#-5}$$

$$m_j^{(i)} := deg(e_j^{(i)}) \qquad \delta_{k,j}^{(i)} := m_j^{(i)} - m_k^{(i-1)} \tag{\#-6}$$

$$\varphi_i(e_j^{(i)}) = \sum_{k=1}^{\sigma(i-1)} M_{k,j}^{(i)}\, e_k^{(i-1)} \qquad M_{k,j}^{(i)} \in H^0(P, O_P(\delta_{k,j}^{(i)})). \tag{\#-7}$$

Here we make a remark that the minimality of $\mathbb{F}_\bullet$ implies that $M^{(i)}_{k,j}=0$ if $\delta^{(i)}_{k,j}\leq 0$.

For an integer $k\geq 1$, we set a graded S-module $\mathbb{Z}^{(k)}_X$ to be $Im(\varphi_k)$ and call it *the k-th syzygy module* of X. Also we set a coherent sheaf $Z^{(k)}_X$ to be $(\mathbb{Z}^{(k)}_X)^{(\sim)}$ and call it *the sheaf of k-th syzygy module* of X. Obviously, we see that the sheaf of ideals I_X which define $j(X)$ in P coincides with the sheaf of 1-st syzygy module of X. In general, we have always $\mathbb{Z}^{(k)}_X=\Gamma_*(Z^{(k)}_X)$ since $depth_{S_+}(R_X)\geq 1$. For more information on the geometric properties of the sheaves $Z^{(k)}_X$, see [15].

Let us take the de Rham complex $\Omega^\bullet_P$ of $P=\mathbb{P}^N(\mathbb{C})$:

$$0\longrightarrow O_P\xrightarrow{d}\Omega^1_P\xrightarrow{d}\Omega^2_P\xrightarrow{d}\dots\xrightarrow{d}\Omega^N_P\longrightarrow 0$$

and the ideal order filtration *(cf. [5])* $F^q_\nu\Omega^\bullet_P$:

$$\begin{array}{l}0\longrightarrow I_X^{\nu+q}\xrightarrow{d}I_X^{\nu+q-1}\Omega^1_P\xrightarrow{d}\dots\\ \xrightarrow{d}I_X^{\nu+1}\Omega^{q-1}_P\xrightarrow{d}\Omega^q_P\xrightarrow{d}\dots\xrightarrow{d}\Omega^N_P\longrightarrow 0.\end{array}$$

Now we fix ν and see $Gr^q_{F^\nu}(\Omega^\bullet_P)=F^q_\nu/F^{q+1}_\nu$:

$$\begin{array}{l}0\longrightarrow I_X^{\nu+q}/I_X^{\nu+q+1}\xrightarrow{\overline{d_I}}I_X^{\nu+q-1}/I_X^{\nu+q}\otimes\Omega^1_P\xrightarrow{\overline{d_I}}\dots\\ \xrightarrow{\overline{d_I}}I_X^{\nu+1}/I_X^{\nu+2}\otimes\Omega^{q-1}_P\xrightarrow{\overline{d_I}}\Omega^q_P|_{X_{(\nu)}}\longrightarrow 0,\end{array}$$

where $X_{(\nu)}=(|X|,O_P/I_X^{\nu+1})$. Contrary to the fact that the exterior derivative d is not O_P-linear, the map $\overline{d_I}$ is O_P-linear and compatible with tensoring by $O_P(m)$. In case $\nu=0$, we have also an O_P-linear composition homomorphism :

$$(\#\text{-}8)\qquad d_I=\overline{d_I}\circ r:\quad I_X(m)\otimes\Omega^{q-1}_P\xrightarrow{r}I_X/I_X^2(m)\otimes\Omega^{q-1}_P\xrightarrow{\overline{d_I}}\Omega^q_P|_X(m),$$

where the homomorphism r is the canonical one.

Next we consider the following exact commutative diagram including the global lifting sequence (LFT) and the 1-st infinitesimal lifting sequence $(\overline{LFT})$:

$$(\#\text{-}9)\qquad \begin{array}{ccccccccc}0&\longrightarrow&I_X\otimes\Omega^q_P(m)&\xrightarrow{\alpha_{LFT}}&\Omega^q_P(m)&\xrightarrow{\beta_{LFT}}&\Omega^q_P(m)|_X&\longrightarrow&0\\ &&{\scriptstyle r}\downarrow&&\downarrow&&\|&&\\ 0&\longrightarrow&N^\vee_{X/P}\otimes\Omega^q_P(m)&\xrightarrow{\overline{\alpha_{LFT}}}&\Omega^q_P(m)|_{X_{(1)}}&\xrightarrow{\overline{\beta_{LFT}}}&\Omega^q_P(m)|_X&\longrightarrow&0,\end{array}$$

which induces obstruction maps (for the 1-st infinitesimal lifting and the global lifting):

$$(\#\text{-}10)\qquad \begin{array}{ccc}H^0(X,\Omega^q_P(m)|_X)&\xrightarrow{\delta_{LFT}}&H^1(P,I_X\otimes\Omega^q_P(m))\\ \|&&\downarrow{\scriptstyle r}\\ H^0(X,\Omega^q_P(m)|_X)&\xrightarrow[\overline{\delta_{LFT}}]{}&H^1(X,N^\vee_{X/P}\otimes\Omega^q_P(m)).\end{array}$$

Then we define a map:

(#-11) $$\widehat{L}_X = \overline{\delta_{LFT}} \circ \overline{d_I} : H^0(X, N_X^\vee(m) \otimes \Omega_P^{q-1}) \to H^1(X, N_{X/P}^\vee(m) \otimes \Omega_P^q),$$

which is called the *meta-Lefschetz operator* with respect to the projective embedding $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. In general, this operator does not coincide with a coupling by the 1-st Chern class of a line bundle up to constants (cf. [7]). For more general definition and properties on meta-Lefschetz operators, see [13].

Putting $V := H^0(P, O_P(1)) = \oplus_{i=0}^N \mathbb{C}Z_i$, let us consider the Koszul complex $\mathbb{K}_\bullet$ for S/S_+ as an S-module:

(#-12)
$$(S/S_+ \xleftarrow{\kappa_0})K_0 = S \xleftarrow{\kappa_1} K_1 = S \otimes V \xleftarrow{\kappa_2} \cdots \xleftarrow{\kappa_{N+1}} K_{N+1} = S \otimes \bigwedge^{N+1} V \longleftarrow 0.$$

In the sequel, to use familiar operations on differential forms the element Z_i in V will be denoted by dZ_i symbolically. This has no spectial geometric meaning on $P = \mathbb{P}^N(\mathbb{C})$ since Z_i is only a section of a line bundle and not a function on P.

The well-known fact from Spectral sequence theory tells us that the three complexes $\mathbb{F}_\bullet \otimes (S/S_+)$, $\mathbb{K}_\bullet \otimes (R_X)$, and the total complex $Tot(\mathbb{F}_\bullet \otimes \mathbb{K}_\bullet)$ of the double complex $\mathbb{F}_\bullet \otimes \mathbb{K}_\bullet$ are quasi-isomorphic with each others (in the derived category of S-modules) and give the same homology $Tor_\bullet^S(R_X, S/S_+)$. By the minimality of the complex $\mathbb{F}_\bullet$ and considering the short exact sequence: $\mathbb{F}_{q+1} \to \mathbb{F}_q \to \mathbb{Z}_X^{(q)} \to 0$, the differential maps of the complex $\mathbb{F}_\bullet \otimes (S/S_+)$ is zero and we have the isomorphisms : $(\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)}) \cong F_q \otimes S/S_+ = H_q(\mathbb{F}_\bullet \otimes (S/S_+)) \cong H_q(Tot(\mathbb{F}_\bullet \otimes \mathbb{K}_\bullet)) \cong H_q(\mathbb{K}_\bullet \otimes (R_X))$, which give $Tor_q^S(R_X, S/S_+)$. The composed isomorphism :

(#-13) $$\lambda : (\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)}) \xrightarrow{\sim} H_q(\mathbb{K}_\bullet \otimes (R_X))$$

forms a part of our isomorphism : $\overline{\rho^{(q,m)}} : (\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)})_{(m)} \cong Tor_q^S(R_X, S/S_+)_{(m)} \tilde{\to} Im[\overline{\delta}_{LFT}] \subseteq H^1(N^\vee(m) \otimes \Omega_P^q)$, which will be given in Theorem 1.12. To get the isomorphism λ, we use the following key diagram which compares the complex $Tot(\mathbb{F}_\bullet \otimes \mathbb{K}_\bullet)$ with the bordered complexes $\mathbb{F}_\bullet \otimes (S/S_+)$ and $\mathbb{K}_\bullet \otimes (R_X)$.

(#-14)
$$\begin{tikzcd}
 & & & {} \arrow[r, "\kappa_1"] \arrow[d, "-\varphi_q"'] & F_q \arrow[r, "\kappa_0"] \arrow[d, "\varphi_q"] & F_q \otimes (S/S_+) \arrow[d, "0-map"] \\
 & & & F_{q-1} \otimes V \arrow[r, "\kappa_1"] \arrow[d, "-\varphi_{q-1}"'] & F_{q-1} \arrow[r, "\kappa_0"] \arrow[d, "\varphi_{q-1}"] & F_{q-1} \otimes (S/S_+) \arrow[d] \\
 & & & \vdots \arrow[d, "-\varphi_2"'] & \vdots \arrow[d, "\varphi_2"] & (\mathrm{b}d.cpx.) \arrow[d] \\
{} \arrow[r, "\kappa_q"] \arrow[d, "(-1)^q\varphi_1"'] & F_1 \otimes \bigwedge^{q-1} V \arrow[r, "\kappa_{q-1}"] \arrow[d, "(-1)^{q-1}\varphi_1"] & \cdots \arrow[r, "\kappa_2"] & F_1 \otimes V \arrow[r, "\kappa_1"] \arrow[d, "-\varphi_1"'] & F_1 \arrow[d, "\varphi_1"] & {} \\
F_0 \otimes \bigwedge^q V \arrow[r, "\kappa_q"] \arrow[d, "\varphi_0"'] & F_0 \otimes \bigwedge^{q-1} V \arrow[r, "\kappa_{q-1}"] \arrow[d, "\varphi_0"] & \cdots \arrow[r, "\kappa_2"] & F_0 \otimes V \arrow[r, "\kappa_1"] & F_0 & \\
R_X \otimes \bigwedge^q V \arrow[r, "\overline{\kappa_q}"] & R_X \otimes \bigwedge^{q-1} V \arrow[r, "\overline{\kappa_{q-1}}"] & (\mathrm{b}d.cpx.) \arrow[r] & {} & &
\end{tikzcd}$$

Since $H_q(\mathbb{K}_\bullet \otimes (R_X))$ is a quotient of $Ker[\overline{\kappa_q} : R_X \otimes \bigwedge^q V \to R_X \otimes \bigwedge^{q-1} V] \cong \oplus_{m\geq q} H^0(P, \Omega_P^q|_X(m))$, for each integer $m \geq q \geq 1$ (N.B. otherwise $\Gamma(P, Z_X^{(q)}(m)) = 0$), we will construct a homomorphism $\widetilde{\lambda} : \Gamma(P, Z_X^{(q)}(m)) \to H^0(P, I_X \otimes \Omega_P^{q-1}(m))$ which makes the diagram:

$$\text{(\#-15)} \qquad \begin{array}{ccc} \Gamma(P, Z_X^{(q)}(m)) & \xrightarrow{\ \widetilde{\lambda}\ } & H^0(P, I_X \otimes \Omega_P^{q-1}(m)) \\ {\scriptstyle can.}\downarrow & & \downarrow{\scriptstyle (can.)\circ(\frac{1}{m})d_I} \\ (\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)}) & \underset{\cong}{\xrightarrow{\ \lambda\ }} & H_q(\mathbb{K}_\bullet \otimes (R_X)), \end{array}$$

commutative, where the homomorphism d_I is given at (#-8).

Remark 1.2 *The correspondence by the map λ in the diagram (#-15) is described as follows. Let us take $\overline{z} \in (\mathbb{Z}_X^{(q)}/S_+ \cdot \mathbb{Z}_X^{(q)})_{(m)}$, $\overline{k} \in H_q(\mathbb{K}_\bullet \otimes (R_X))_{(m)}$ and their representatives : $a_0 \in (\mathbb{Z}_X^{(q)})_{(m)}$ of $\overline{z}$ and $a_q \in (K_q \otimes (R_X))_{(m)}$ of $\overline{k}$, respectively. Then the equality $\lambda(\overline{z}) = \overline{k}$ holds if and only if there exists a system of elements $\{a_i \in (F_{q-1-i} \otimes \bigwedge^i V)_{(m)} | i = 1, 2, \ldots, q-1\}$ and $\{b_j \in (F_{q-j} \otimes \bigwedge^j V)_{(m)} | j = 1, 2, \ldots, q\}$ with the condition: $a_i = \kappa_{i+1}(b_{i+1})$ $\quad (i = 0, \ldots, q-2)$ and $a_i = (-1)^i \varphi_{q-i}(b_i)$ $\quad (i = 1, \ldots, q-1)$ $a_q = \varphi_0(b_q)$ in the diagram (#-14). The crucial point is that, even if we fix the elements $\overline{z}$ and $\overline{k}$ with $\lambda(\overline{z}) = \overline{k}$, the choice of such a system $\{a_i\}_{i=1}^{q-1}$ and $\{b_j\}_{j=1}^{q}$ is <u>not unique</u> as elements in general but is unique up to boundaries.*

To construct the homomorphism $\widetilde{\lambda}$, let us review several definitions in our previous papers. First we refer [14] and recall the Ogus complex $(\Sigma_{H/W/B}^\bullet(m), \nabla_{OG} = \nabla_{[m]OG}^{(\bullet)})$ for a smooth quasi-projective morphism $f : W \to B$ of pure relative dimension between algebraic schemes W and B over $\mathbb{C}$ and for a line bundle H over W, where m denotes a fixed integer which is called *the twist degree of the Ogus complex*. A local section of $\Sigma_{H/W/B}^q(m)$ will be called *(Ogus) pseudo q-form* of twist degree m.

Lemma 1.3 (cf. [14]) *The Ogus complex $(\Sigma_{H/W/B}^\bullet(m), \nabla_{OG} = \nabla_{[m]OG}^{(\bullet)})$ has also Koszul operators $\kappa = \kappa^{(q)} : \Sigma_{H/W/B}^q(m) \to \Sigma_{H/W/B}^{q-1}(m)$ and the pair $(\Sigma_{H/W/B}^\bullet(m), \kappa)$ forms also a complex of sheaves, which is called* Koszul-Ogus complex. *Moreover, we have*

$$\text{(\#-16)} \qquad \Sigma_{H/W/B}^q(m) \cong (\bigwedge^q \Sigma_{H/W/B}^1) \otimes H^m, \qquad \Sigma_{H/W/B}^0(m) \cong H^m$$

$$\text{(\#-17)} \qquad \kappa^{(q+1)} \circ \nabla_{[m]OG}^{(q)} + \nabla_{[m]OG}^{(q-1)} \circ \kappa^{(q)} = m \cdot I_d,$$

which gives a chain homotpy between 0 and $m \cdot I_d$.

For any local sections $\sigma_1 \in \Sigma_{H/W/B}^{q_1}(m_1)$ and $\sigma_2 \in \Sigma_{H/W/B}^{q_2}(m_2)$, then $\sigma_1 \wedge \sigma_2 \in \Sigma_{H/W/B}^{q_1+q_2}(m_1 + m_2)$ and

$$\text{(\#-18)} \qquad \nabla_{OG}(\sigma_1 \wedge \sigma_2) = \nabla_{OG}(\sigma_1) \wedge \sigma_2 + (-1)^{q_1} \sigma_1 \wedge \nabla_{OG}(\sigma_2)$$

$$\text{(\#-19)} \qquad \kappa(\sigma_1 \wedge \sigma_2) = \kappa(\sigma_1) \wedge \sigma_2 + (-1)^{q_1} \sigma_1 \wedge \kappa(\sigma_2)$$

In particular, if the local sections satisfy : $\sigma_1 \in \Sigma_{H/W/B}^0(m_1)$, $\sigma_2 \in \Sigma_{H/W/B}^q(m_2)$, and $\nabla_{OG}(\sigma_2) = 0$, then $\sigma_1 \cdot \sigma_2 \in \Sigma_{H/W/B}^q(m_1 + m_2)$ and $\nabla_{OG}(\sigma_1 \cdot \sigma_2) = \nabla_{OG}(\sigma_1) \wedge \sigma_2$.

Remark 1.4 *For example, we see from Lemma (1.3) above that* $\Sigma^q_{H/W/B}(m+1) \cong (\Sigma^q_{H/W/B}(m)) \otimes_{O_W} H$. *However, the Ogus derivation is really a differential operator of order* 1 *and the* O_W*-linear tensor produt of the homomorphisms of sheaves* $\nabla_{[m]OG}$ *and* 1_H *is not well-defined. The crucial point of the Ogus complex is that the derivation* $\nabla_{[m+1]OG}$ *is still well-defined even in this situation. Thus, fixing the Ogus complex* $(\Sigma^\bullet_{H/W/B}, \nabla_{OG} = \nabla^{(\bullet)}_{[m]OG})$ *for the line bundle* H *with twist degree zero, if we take another line bundle* $\widetilde{H}$*, then there is* no *guarantee that we can make* $\{\Sigma^\bullet_{H/W/B} \otimes \widetilde{H}^m\}$ *into a (co-chain) complex with defining a suitable derivations, in general.*

Using this Ogus complex, at the level of sheaves, we can replace with reversing the arrows the Koszul operations $\kappa_\bullet$ of the Koszul complex (#-12) by exterior derivations and get the complex:

$$(\#\text{-}20) \qquad K_0 = S \xrightarrow{d_{EX}} K_1 = S \otimes V \xrightarrow{d_{EX}} \cdots \xrightarrow{d_{EX}} K_{N+1} = S \otimes \bigwedge^{N+1} V \longrightarrow 0.$$

Since the local expression of the exterior derivation itself on $\mathbb{P}^N(\mathbb{C})$ has non-trivial expression as we will see in the sequel, we need to introduce the Ogus derivation.

In case of $W = P = \mathbb{P}^N(\mathbb{C})$, $H = O_P(1)$, and $B = \mathrm{Spec}(\mathbb{C})$, the system of homogeneous coordinates $Z_0, \ldots, Z_N \in H^0(P, O_P(1)) = V$ given above can separate two points including infinitesimally near points, which implies the isomorphism:

$$(\#\text{-}21) \qquad \Sigma^1_{O_P(1)/P/\mathbb{C}}(1) \cong J^1_{P/\mathbb{C}}(H) \cong \bigoplus_{a=0}^{N} O_P\, j^1(Z_a),$$

where $J^1_{P/\mathbb{C}}(H)$ and $j^1 : H \to J^1_{P/\mathbb{C}}(H)$ denote the 1-jet sheaf (the 1-st principal parts) of the line bundle $H = O_P(1)$ and the 1-jet map, respectively. To see the correspondence between the complex (#-20) and the Ogus complex $(\Sigma^\bullet_{H/W/B}(m), \nabla_{OG})$, we replace $j^1(Z_a)$ in (#-21) by dZ_a, which is only a symbol and has no geometric meaning.

Then we have:

$$(\#\text{-}22) \qquad \Sigma^q_{O_P(1)/P/\mathbb{C}}(m) \cong \bigoplus_{0 \le a_1 < \ldots < a_q \le N} O_P(m-q)\, dZ_{a_1} \wedge \ldots \wedge dZ_{a_q}.$$

Lemma 1.5 (cf. [14]) *For* $W = P = \mathbb{P}^N(\mathbb{C})$, $H = O_P(1)$, *and* $B = \mathit{Spec}(\mathbb{C})$, *the local expressions of the Ogus derivation* $\nabla_{OG} : \Sigma^q_{O_P(1)/P/\mathbb{C}}(m) \cong \oplus O_P(m-q)\, dZ_{a_1} \wedge \ldots \wedge dZ_{a_q} \to \Sigma^{q+1}_{O_P(1)/P/\mathbb{C}}(m) \cong \oplus O_P(m-q-1)\, dZ_{b_0} \wedge \ldots \wedge dZ_{b_q}$ *and the Koszul operator (for the Ogus complex)* $\kappa : \Sigma^q_{O_P(1)/P/\mathbb{C}}(m) \cong \oplus O_P(m-q)\, dZ_{a_1} \wedge \ldots \wedge dZ_{a_q} \to \Sigma^{q-1}_{O_P(1)/P/\mathbb{C}}(m) \cong \oplus O_P(m-q+1)\, dZ_{b_1} \wedge \ldots \wedge dZ_{b_{q-1}}$ *on an openset* $U \subseteq U_k = D_+(Z_k)$ *are given as follows.*

(#-23)

$$\begin{aligned}\nabla_{OG}(f \otimes Z_k^{m-q} \quad & dZ_{a_1} \wedge \ldots \wedge Z_{a_q}) \\ &= \left\{(m-q) \cdot f - \sum_{r=0, r\neq k}^{N} \left(\frac{Z_r}{Z_k}\right) \frac{\partial f}{\partial(Z_r/Z_k)}\right\} \cdot dZ_k \wedge dZ_{a_1} \wedge \ldots \wedge dZ_{a_q} \otimes Z_k^{m-q-1} \\ &\qquad + \sum_{r=0, r\neq k}^{N} \frac{\partial f}{\partial(Z_r/Z_k)} \cdot dZ_r \wedge dZ_{a_1} \wedge \ldots \wedge dZ_{a_q} \otimes Z_k^{m-q-1}\end{aligned}$$

$$(\#\text{-}24)\quad \kappa(f\otimes Z_k^{m-q}\, dZ_{a_1}\wedge\ldots\wedge dZ_{a_q}) = \sum_{j=1}^{q}(-1)^{j-1} f\cdot(\frac{Z_{a_j}}{Z_k})\otimes Z_k^{m-q+1}\, dZ_{a_1}\wedge\ldots\overset{j}{\smile}\ldots\wedge dZ_{a_q}$$

Corollary 1.6 *For a local section* $G = g\otimes Z_k^m \in O_P(m) = \Sigma^0_{O_P(1)/P/\mathbb{C}}(m)$ *on an openset* $U\subseteq D_+(Z_k)$,

(#-25)

$$\begin{aligned}\nabla_{OG}(G) &= \nabla_{OG}(g\otimes Z_k^m)\\ &= \left\{m\cdot g - \sum_{r=0,r\neq k}^{N}\left(\frac{Z_r}{Z_k}\right)\frac{\partial\, g}{\partial(Z_r/Z_k)}\right\}dZ_k\otimes Z_k^{m-1} + \sum_{r=0,r\neq k}^{N}\frac{\partial\, g}{\partial(Z_r/Z_k)}\cdot dZ_r\otimes Z_k^{m-1}\end{aligned}$$

Remark 1.7 *In particular, for a homogeneous polynomial* F *of degree* $m-q$, *putting* $f := F/(Z_k)^{m-q}$, *we have*

$$\begin{aligned}\nabla_{OG}(F\, dZ_{a_1}\wedge\ldots\wedge dZ_{a_q}) &= \sum_{r=0}^{N}\frac{\partial F}{\partial Z_r}\cdot dZ_r\wedge dZ_{a_1}\wedge\ldots\wedge dZ_{a_q}\\ \kappa(F\, dZ_{a_1}\wedge\ldots\wedge dZ_{a_q}) &= \sum_{j=1}^{q}(-1)^{j-1}Z_{a_j}\cdot F\, dZ_{a_1}\wedge\ldots\overset{j}{\smile}\ldots\wedge dZ_{a_q}\end{aligned}$$

which shows that the Ogus derivation ∇_{OG} *and the Koszul operator* κ *of the Ogus complex coincide with the exterior derivation of the polynomial ring* S *and the (usual) Koszul operator for the maximal ideal* $S_+ = (Z_0,\ldots,Z_N)$ *of the polynomial ring* S *at the global level.*

Definition 1.8 *For (a general) Ogus complex* $(\Sigma^\bullet_{H/W/B}(m),\nabla_{OG})$, *we define* the revised Ogus derivation $\widehat{\nabla}$ *as follows.*

$$(\#\text{-}26)\qquad \widehat{\nabla} = \begin{cases}\frac{1}{m}\cdot\nabla_{OG} & (if\quad m\neq 0)\\ \nabla_{OG} & (if\quad m=0)\end{cases}$$

— * —

Let us return to the previous diagram (#-14) and its core part $Tot(\mathbb{F}_\bullet\otimes\mathbb{K}_\bullet)$. Recall (#-6) and that

$$(\#\text{-}27)\qquad F_t\otimes K_s = \bigoplus_{u=1}^{\sigma(t)}\ \bigoplus_{0\le j_1<\ldots<j_s\le N} S\quad e_u^{(t)}\, dZ_{j_1}\wedge\ldots\wedge dZ_{j_s}\quad .$$

Then, under the assumption: m an integer, $W = P = \mathbb{P}^N(\mathbb{C})$, $H = O_P(1)$, and $B = \mathrm{Spec}(\mathbb{C})$, we set:

$$(\#\text{-}28)\qquad \begin{aligned}\Xi^{s,t}_{X/P}(m) &= \bigoplus_{u=1}^{\sigma(t)}\ \Sigma^s_{O_P(1)/P/\mathbb{C}}\,(m-m_u^{(t)})\, e_u^{(t)}\\ &= \bigoplus_{u=1}^{\sigma(t)}\ \bigoplus_{0\le j_1<\ldots<j_s\le N} O_P(m-m_u^{(t)}-s)\quad e_u^{(t)}\, dZ_{j_1}\wedge\ldots\wedge dZ_{j_s}\end{aligned}$$

$$
(\#\text{-}29)\qquad
\begin{array}{llcl}
\partial_I = \partial_I^{s,t} = \oplus\kappa : & \Xi^{s,t}_{X/P}(m) & \longrightarrow & \Xi^{s-1,t}_{X/P}(m) \\
\partial_{II} = \partial_{II}^{s,t} = (-1)^s\varphi_t : & \Xi^{s,t}_{X/P}(m) & \longrightarrow & \Xi^{s,t-1}_{X/P}(m) \\
\delta_I = \delta_I^{s,t} = \oplus\widehat{\nabla} : & \Xi^{s,t}_{X/P}(m) & \longrightarrow & \Xi^{s+1,t}_{X/P}(m) \quad ,
\end{array}
$$

where, for a local section $\psi = \sum_{u=1}^{\sigma(t)} \psi_u e_u^{(t)} \in \Gamma(U, \Xi^{s,t}_{X/P}(m))$; $\psi_u \in \Gamma(U, \Sigma^s_{O_P(1)/P/\mathbb{C}}(m - m_u^{(t)}))$, using $\{M^{(t)}_{w,u}\}$ of (#-7), the O_P-linear homomorphisms ∂_I, ∂_{II} as boundary operators, and the abelian sheaf homomorphism δ_I as coboundary operator are defined as follows.

$$
(\#\text{-}30)\qquad \partial_I(\psi) := \sum_{u=1}^{\sigma(t)} \kappa(\psi_u)\, e_u^{(t)}
$$

$$
(\#\text{-}31)\qquad \partial_{II}(\psi) := \sum_{w=1}^{\sigma(t-1)} (-1)^s \left(\sum_{u=1}^{\sigma(t)} M^{(t)}_{w,u} \cdot \psi_u \right) e_w^{(t-1)}
$$

$$
(\#\text{-}32)\qquad \delta_I(\psi) := \sum_{u=1}^{\sigma(t)} \widehat{\nabla}(\psi_u)\, e_u^{(t)}
$$

Since the Koszul operator κ and the revised Ogus derivation $\widehat{\nabla}$ keep the twist degree of the Ogus complex by the line bundle $H = O_P(1)$, the well-definedness of the boundary operator ∂_I and coboundary operator δ_I are rather obvious. Now we consider the the well-definedness of the boundary operator ∂_{II}. Using the fact that $M^{(t)}_{w,u} \in \Gamma(P, O_P(\delta^{(t)}_{w,u}))$, $\psi_u \in \Gamma(U, \Sigma^s_{O_P(1)/P/\mathbb{C}}(m - m_u^{(t)}))$, and (#-6), we see that $m - m_u^{(t)} + \delta^{(t)}_{w,u} = m - m_u^{(t)} + m_u^{(t)} - m_w^{(t-1)} = m - m_w^{(t-1)}$, and $(-1)^s(\sum_{u=1}^{\sigma(t)} M^{(t)}_{w,u} \cdot \psi_u) \in \Gamma(U, \Sigma^s_{O_P(1)/P/\mathbb{C}}(m - m_w^{(t-1)}))$, or equivalently $\partial_{II}(\psi) \in \Gamma(U, \Xi^{s,t-1}_{X/P}(m))$.

From the construction above, we obviously obtain the following statement.

Lemma 1.9 *We have $\partial_I \circ \partial_I = \partial_{II} \circ \partial_{II} = \partial_I \circ \partial_{II} + \partial_{II} \circ \partial_I = 0$. Let us set that $\partial := \partial_I + \partial_{II}$ and*

$$
\Xi^{(n)}_{X/P}(m) := \bigoplus_{s+t=n} \Xi^{s,t}_{X/P}(m).
$$

Then the pair $(\Xi^{(\bullet)}_{X/P}(m), \partial)$ forms a sheaf of complex and the complex $(\Gamma_*(\Xi^{(\bullet)}_{X/P}(m)), \partial)$ of S-modules coincides with the total complex $Tot(\mathbb{F}_\bullet \otimes \mathbb{K}_\bullet)$ in the key diagram (#-14). In particular, $\Gamma_*(\Xi^{s,t}_{X/P}) \cong F_t \otimes \bigwedge^s V$ naturally and the bounday operators $\partial_I^{s,t}$ and $\partial_{II}^{s,t}$ correspond to κ_s and $(-1)^s\varphi_t$ of (#-14).

Next we define the m-test projection which is a projection from $\Xi^{s,t}_{X/P}(m)$ to its direct summands whose corresponding basis satisfy $m_u^{(t)} = deg(e_u^{(t)}) \geq m$, or equivalently whose twist degrees are non-positive.

Definition 1.10 *The projection* $pr_{\geq m} = pr^{s,t}_{\geq m}$*:*

(#-33)

$$pr^{s,t}_{\geq m} : \Xi^{s,t}_{X/P}(m) = \bigoplus_{u=1}^{\sigma(t)} \Sigma^s_{O_P(1)/P/\mathbb{C}}\,(m - m^{(t)}_u)\; e^{(t)}_u \to \bigoplus_{\substack{1 \leq u \leq \sigma(t) \\ m_u \geq m}} \Sigma^s_{O_P(1)/P/\mathbb{C}}(m - m^{(t)}_u)\; e^{(t)}_u \subseteq \Xi^{s,t}_{X/P}(m),$$

is called the m-test projection.

Lemma 1.11 *Assume that a local section* $\psi \in \Gamma(U, \Xi^{s,t}_{X/P}(m))$ *satisfies :* $\partial_I(\psi) = 0$ *and* $pr_{\geq m}(\psi) = 0$ *for the m-test projection. Then* $\partial_I \circ \delta_I(\psi) = \psi$*, namely* $\delta_I(\psi) \in (\partial_I)^{-1}(\psi) \subseteq \Gamma(U, \Xi^{s+1,t}_{X/P}(m))$*. Moreover,* $\partial_{II} \circ \delta_I(\psi) \in \Gamma(U, \Xi^{s+1,t-1}_{X/P}(m))$ *satisfies* $pr_{\geq m}(\partial_{II} \circ \delta_I(\psi)) = 0$*. Further more, if* $\partial_{II}(\psi) = 0$*, then* $\partial_I(\partial_{II} \circ \delta_I(\psi)) = 0$ *and* $\partial_{II}(\partial_{II} \circ \delta_I(\psi)) = 0$*.*

Proof. Since the boundary operators ∂_I and the coboundary operator δ_I act on each direct summand separately, the first claim $\partial_I \circ \delta_I(\psi) = \psi$ is obvious from (#-17). On the latter claim: $pr_{\geq m}(\partial_{II} \circ \delta_I(\psi)) = 0$, let us assume that $pr_{\geq m}(\partial_{II} \circ \delta_I(\psi)) \neq 0$. Since the boundary operator ∂_{II} raises strictly the twist degree of the Ogus complex by the small remark on $\{M^{(i)}_{k,j}\}$ after (#-7) and the coboundary operator δ_I keep the twist degree, we see easily that $pr_{\geq m}(\psi) \neq 0$, which is a contradiction. The remainder is trivial. ∎

Now we come to the stage where we can construct the homomorphism $\widetilde{\lambda}$ which fits into the diagram (#-15). By the Lemma (1.9), the complex :

$$0 \longrightarrow Ker(\partial^{0,q-1}_{II}) \xrightarrow{incl.} \Xi^{0,q-1}_{X/P} \xrightarrow{\partial^{0,q-1}_{II}} \Xi^{0,q-2}_{X/P} \xrightarrow{\partial^{0,q-2}_{II}} \cdots$$

(#-34)

$$\xrightarrow[\partial^{0,2}_{II}]{} \Xi^{0,1}_{X/P} \xrightarrow[\partial^{0,1}_{II}]{} \Xi^{0,0}_{X/P} = O_P \xrightarrow[can.]{} O_X \longrightarrow 0$$

is exact. Taking their $\Gamma_*(P, -)$, we obtain a truncation of the complex of S-modules (#-4), which means that $Ker(\partial^{0,q-1}_{II}) \cong Z^{(q)}_X$. Since the resolution (#-4) is minimal, we may assume $1 \leq q \leq h$ and $m \geq q$, otherwise $\Gamma(P, Z^{(q)}_X(m)) = 0$ and we do not have to describe the syzygy class map explicitly.

Now we define:

$$\text{(\#-35)} \qquad \widetilde{\lambda} := \prod_{u=0}^{q-2} (\partial^{u+1,q-1-u}_{II} \circ \delta^{u,q-1-u}_I) : \; \Xi^{0,q-1}_{X/P}(m) \longrightarrow \Xi^{q-1,0}_{X/P}(m).$$

Return to the fact that $Z^{(q)}_X(m) \subseteq \Xi^{0,q-1}_{X/P}(m) = \bigoplus_{u=1}^{\sigma(q-1)} O_P\,(m - m^{(q-1)}_u)\; e^{(q-1)}_u$, for a global section $\psi \in \Gamma(P, Z^{(q)}_X(m))$ and the m-test projection $pr_{\geq m}$ of $\Xi^{0,q-1}_{X/P}(m)$, we have $pr_{\geq m}(\psi) = 0$. Applying Lemma(1.11) inductively, we see that $\widetilde{\lambda}(\Gamma(P, Z^{(q)}_X(m))) \subseteq \Gamma(P, Ker(\partial^{q-1,0}_I)) \cap \partial^{q-1,1}_{II}(\Gamma(P, \Xi^{q-1,1}_{X/P}(m))) \subseteq H^0(P, I_X \otimes \Omega^{q-1}_P(m))$.

To show that this homomorphism $\widetilde{\lambda}$ of abelian sheaves make the diagram (#-15) commutative, let us recall Remark (1.2). Suppose that an element $a_0 \in \Gamma(P, Z_X^{(q)}(m))$ is given. For integers $i = 1, \dots, q-1$ and $j = 1, \dots, q$, we put

$$(\#\text{-}36) \qquad a_i := \prod_{u=0}^{i-1} (\partial_{II}^{u+1, q-1-u} \circ \delta_I^{u, q-1-u})(a_0), \quad b_j := \delta_I^{j-1, q-j}(a_{j-1}), \quad a_q := \varphi_0(b_q).$$

Then we obtain a system of elements $\{a_i\}_{i=0}^q$ and $\{b_j\}_{j=1}^q$ as in Remark (1.2). By the definition of $\widetilde{\lambda}$ above, we get $a_{q-1} = \widetilde{\lambda}(a_0)$. Thus we have only to show that

$$(\#\text{-}37) \qquad a_{q-1} \in \Gamma(P, I_X \otimes \Omega_P^{q-1}(m)) \qquad a_q = \left(\frac{1}{m}\right) \cdot d_I(a_{q-1}) \in \Gamma(P, \Omega_P^q|_X(m)).$$

Let us consider precisely the process of getting a_q from $a_{q-1} \in \Gamma(P, \Xi_{X/P}^{q-1,0}(m))$, where the space $\Gamma(P, \Xi_{X/P}^{q-1,0}(m))$, namely $\Gamma(P, \Sigma_P^{q-1}(m))$ corresponds $S \otimes \wedge^{q-1} V$ in the diagram (#-14). The element $b_q = \delta_I(a_{q-1}) \in \Gamma(P, \Xi_{X/P}^{q,0}(m))$ is written as $b_q = \widehat{\nabla}_{OG}(a_{q-1}) = (\frac{1}{m}) \cdot \nabla_{OG}(a_{q-1}) \in \Gamma(P, \Sigma_P^q(m))$. Then, the canonical image $a_q \in R_X \otimes \wedge^q V$ of b_q can be consider as the element $a_q \in \Gamma(P, \Sigma_P^q(m) \otimes O_X)$ which is the image of b_q by the canonical homomorphism $\Sigma_P^q(m) \to \Sigma_P^q(m) \otimes O_X$. On the other hand, since $a_{q-1} = \partial_{II}(b_{q-1}) \in Im[\partial_{II} : \Xi_{X/P}^{q-1,1}(m) \to \Xi_{X/P}^{q-1,0}(m)]$, we know that $a_{q-1} \in \Gamma(P, I_X \otimes \Sigma_P^{q-1}(m))$. Now we define a homomorphism of abelian sheaves:

$$\overline{\nabla}_{EN} := (can.) \circ \nabla_{OG} \circ (incl.) : I_X \otimes \Sigma_P^{q-1}(m) \longrightarrow \Sigma_P^{q-1}(m) \longrightarrow \Sigma_P^q(m) \longrightarrow \Sigma_P^q(m) \otimes O_X.$$

Then we can summarize the argument above and see that $a_q = (\frac{1}{m}) \cdot \overline{\nabla}_{EN}(a_{q-1})$. Moreover, from the construction of the element a_{q-1}, we see that $a_{q-1} \in Ker[\partial_I : \Xi_{X/P}^{q-1,0}(m) \to \Xi_{X/P}^{q-2,0}(m)]$, namely there is an element $\widetilde{a_{q-1}} \in \Gamma(P, I_X \otimes \Omega_P^{q-1}(m))$ such that $\alpha_{(q-1)-EU}(\widetilde{a_{q-1}}) = a_{q-1}$ (on $\alpha_{(q-1)-EU}$, see (#-50) with replacing q by $q-1$). In other words, $\widetilde{a_{q-1}} = a_{q-1}$ in $S \otimes \wedge^{q-1} V$. Now we recall the following diagram from Proposition 2.1 of [14] (N.B. in [14], $\overline{d}_I$ was a misprint of d_I. To avoid needless confusion, we change the previous symbol $\overline{\nabla}_{OG}$ of [14] into the new symbol $\overline{\nabla}_{EN}$ in this paper).

$$(\#\text{-}38) \qquad \begin{array}{ccccccc} 0 \longrightarrow & I_X \otimes \Omega_P^{q-1}(m) & \xrightarrow{\alpha_{(q-1)-EU}} & I_X \otimes \Sigma_P^{q-1}(m) & \xrightarrow{\kappa^{(q-1)}} & I_X \otimes \Sigma_P^{q-2}(m) \\ & \Big\downarrow{\scriptstyle d_I} & & \Big\downarrow{\scriptstyle \overline{\nabla}_{EN}} & & \Big\downarrow{\scriptstyle -\overline{\nabla}_{EN}} \\ 0 \longrightarrow & \Omega_P^q|_X(m) & \xrightarrow{\overline{\alpha_{q-EU}}} & \Sigma_P^q(m) \otimes O_X & \xrightarrow{\overline{\kappa}^{(q)}} & \Sigma_P^{q-1}(m) \otimes O_X. \end{array}$$

Starting from the element $\widetilde{a_{q-1}} \in \Gamma(P, I_X \otimes \Omega_P^{q-1}(m))$, apply the diagram (#-38) above, we see that $\overline{\alpha_{q-EU}}((\frac{1}{m}) \cdot d_I(\widetilde{a_{q-1}})) = (\frac{1}{m})\overline{\nabla}_{EN}(a_{q-1}) = a_q$. Thus we obtain (#-37).

Combining the construction above of $\widetilde{\lambda}$ with our previous results, we have the following theorem on the syzygy class map.

Theorem 1.12 (cf. [13], [14]) *Let X be a projective subscheme of $P = \mathbb{P}^N(\mathbb{C})$ which satisfies arithmetic D_2 condition, namely $R_X = \widetilde{R_X}$. Then, the map $r : H^1(P, I_X \otimes \Omega_P^q(m)) \to H^1(X, N_{X/P}^\vee \otimes \Omega_P^q(m))$ is injective, or equivalently, a global obstruction vanishes if and only if its 1-st infinitesimal obstruction vanishes. In the space $H^1(X, N_{X/P}^\vee(m) \otimes \Omega_P^q)$, all the three subspaces $Im[\overline{\delta_{LFT}} \circ d_I]$, $Im[\widehat{L}_X]$ and the space of infinitesimal obstructions $Im[\overline{\delta_{LFT}}]$ coincide with each others. A composed map :*

(#-39)

$$\rho^{(q,m)} : \Gamma(P, Z_X^{(q)}(m)) \xrightarrow{\ \widetilde{\lambda}\ } H^0(P, I_X \otimes \Omega_P^{q-1}(m))$$

$$\xrightarrow{(\frac{1}{m})d_I} \quad H^0(X, \Omega_P^q(m)|_X) \quad \xrightarrow{\overline{\delta_{LFT}}} Im[\overline{\delta_{LFT}}] \subseteq H^1(X, N_{X/P}^\vee \otimes \Omega_P^q(m))$$

is surjective onto the subspace $Im[\overline{\delta_{LFT}}]$. *Its kernel coincides with the subspace* $\sum_{i=0}^N Z_i \cdot \Gamma(P, Z_X^{(q)}(m-1))$. *Thus the homomorphism* $\rho^{(q,m)}$ *induces an isomorphism* $\overline{\rho^{(q,m)}} : \Gamma_*(P, Z_X^{(q)})_{(m)} \cong Tor_q^S(R_X, S/S_+)_{(m)} \to Im[\overline{\delta_{LFT}}]$. *The map* $\rho^{(q,m)}$ *is called* the q-syzygy class map for degree m part.

Remark 1.13 *To see that, up to* O_P*-linear isomorphisms of* $Z_X^{(q)}$*'s, the map* $\rho^{(q,m)}$ *is independent from the choice of graded minimal free resolutions, it is enough to see the following commutative diagram which show the factorization of the map* $\rho^{(q,m)}$ *into three intrinsic maps : "can."(the canonical quotient map),* λ*, and* $\overline{\delta_{LFT}}$.

(#-40)

$$\begin{tikzcd}
\Gamma(P, Z_X^{(q)}(m)) \arrow[rr, "\rho^{(q,m)}"] \arrow[d, "can."'] & & H^1(X, N_{X/P}^\vee \otimes \Omega_P^q(m)) \\
\Gamma(P, Z_X^{(q)}(m)) \Big/ \sum Z_i \cdot \Gamma(P, Z_X^{(q)}(m-1)) \arrow[rr, "\lambda"'] \arrow[urr, "\overline{\rho^{(q,m)}}"'] & & H^0(X, \Omega_P^q(m)|_X)/Im[H^0(P, \Omega_P^q(m))] \arrow[u, "\overline{\delta_{LFT}}"']
\end{tikzcd}$$

On the map λ *being intrinsic, we can confirm it as follows. The graded minimal free resolution is unique not only up to chain homotopies but also up to chain isomorphisms which can be described completely as is well-known (cf. Lemma 1.3 of [15]). Chain isomorphisms cause* O_P*-linear isomorphisms of* $Z_X^{(q)}$*'s and compatible isomorphisms of the key diagram (#-14), which induce compatible* O_P*-linear isomorphisms of* $Z_X^{(q)}$*'s for the maps* λ*'s.*

§2 The Main Result

Let us recall the resolution (#-4), the coefficients $\{M_{k,j}^{(i)}\}$ in (#-7), and calculate the syzygy class map $\rho^{(q,m)}$ explicitly. We may assume $1 \leq q \leq h$ and $m \geq q$, otherwise $\Gamma(P, Z_X^{(q)}(m)) = 0$. For integres i, j, k, and a with the four conditions: $0 \leq a \leq N$; $1 \leq i \leq h$; $1 \leq k \leq \sigma(i-1)$; $1 \leq j \leq \sigma(i)$, we put:

(#-41)
$$\mu_{k,j}^{(i,a)} := M_{k,j}^{(i)} / Z_a^{\delta_{k,j}^{(i)}} .$$

Now we take a global section ψ of $Z_X^{(q)}(m)$:

(#-42)
$$\psi = \sum_{u=1}^{\sigma(q-1)} \widetilde{M}_u \, e_u^{(q-1)} \in \Gamma(P, Z_X^{(q)}(m)) \subseteq \Gamma(P, \Xi_{X/P}^{0,q-1}(m)) \cong \bigoplus_{u=1}^{\sigma(q-1)} \Gamma(P, O_P(m - m_u^{(q-1)})) \, e_u^{(q-1)} .$$

Then, similarly to (#-6) and (#-41), we define:

$$(\#\text{-}43)\qquad \widetilde{\delta_u} := m - m_u^{(q-1)} \qquad\qquad \widetilde{\mu_u^{(a)}} := \widetilde{M_u}/Z_a^{\widetilde{\delta_u}} \quad .$$

For an integer $u = 1, \ldots, q-1$ and moving indices $t(1), t(2), \cdots, t(u)$ with $1 \leq t(1) \leq \sigma(q-1), \cdots, 1 \leq t(u) \leq \sigma(q-u)$, if $m > m_{t(1)}^{(q-1)} > \cdots > m_{t(u)}^{(q-u)} \geq 1$, we set a positive rational number :

$$(\#\text{-}44)\qquad c^{t(u),\cdots,t(1)} := \frac{1}{m} \cdot \prod_{s=1}^{u} \frac{1}{m - m_{t(s)}^{(q-s)}} \quad \in \mathbb{Q}_{>0} \quad ,$$

otherwise we put $c^{t(u),\cdots,t(1)} = 0$.

Recall Corollary 1.6 and apply the Ogus derivation ∇_{OG} to the sections $M_{k,j}^{(i)} \in \Gamma(P, O_P(\delta_{k,j}^{(i)}))$ and $\widetilde{M_u} \in \Gamma(P, O_P(\widetilde{\delta_u}))$ by regarding the line bundle $O_P(*)$ as the sheaf of Ogus pseudo 0-forms $\Sigma^0_{O_P(1)/P/\mathbb{C}}(*)$. On an open set $U \subseteq D_+(Z_a)$, we have:

(#-45)

$$\begin{aligned} \eta_{k,j}^{(i)} &:= \nabla_{OG}(M_{k,j}^{(i)}) = \nabla_{OG}(\mu_{k,j}^{(i,a)} \otimes Z_a^{\delta_{k,j}^{(i)}}) \\ &= \left\{ \delta_{k,j}^{(i)} \cdot \mu_{k,j}^{(i,a)} - \sum_{r=0, r\neq a}^{N} \left(\frac{Z_r}{Z_a}\right) \frac{\partial\, \mu_{k,j}^{(i,a)}}{\partial(Z_r/Z_a)} \right\} dZ_a \otimes Z_a^{\delta_{k,j}^{(i)}-1} + \sum_{r=0, r\neq a}^{N} \frac{\partial\, \mu_{k,j}^{(i,a)}}{\partial(Z_r/Z_a)} \cdot dZ_r \otimes Z_a^{\delta_{k,j}^{(i)}-1} \end{aligned}$$

(#-46)

$$\begin{aligned} \widetilde{\eta_u} &:= \nabla_{OG}(\widetilde{M_u}) = \nabla_{OG}(\widetilde{\mu_u^{(a)}} \otimes Z_a^{\widetilde{\delta_u}}) \\ &= \left\{ \widetilde{\delta_u} \cdot \widetilde{\mu_u^{(a)}} - \sum_{r=0, r\neq a}^{N} \left(\frac{Z_r}{Z_a}\right) \frac{\partial\, \widetilde{\mu_u^{(a)}}}{\partial(Z_r/Z_a)} \right\} dZ_a \otimes Z_a^{\widetilde{\delta_u}-1} + \sum_{r=0, r\neq a}^{N} \frac{\partial\, \widetilde{\mu_u^{(a)}}}{\partial(Z_r/Z_a)} \cdot dZ_r \otimes Z_a^{\widetilde{\delta_u}-1} \quad . \end{aligned}$$

Then, we see that $\eta_{k,j}^{(i)} \in \Gamma(P, \Sigma^1_{O_P(1)/P/\mathbb{C}}(\delta_{k,j}^{(i)}))$ and $\widetilde{\eta_u} \in \Gamma(P, \Sigma^1_{O_P(1)/P/\mathbb{C}}(\widetilde{\delta_u}))$ with ∇_{OG}-closedness, namely $\nabla_{OG}(\eta_{k,j}^{(i)}) = 0$ and $\nabla_{OG}(\widetilde{\eta_u}) = 0$.

Main Theorem 2.1 (Explicit form of the Syzygy class map) *Under the circumstances, the q-syzygy class map for degree m part:*

$$(\#\text{-}47)\qquad \rho^{(q,m)} : \Gamma(P, Z_X^{(q)}(m)) \longrightarrow H^1(P, N^\vee(m) \otimes \Omega_P^q)$$

sends the section $\psi \in \Gamma(P, Z_X^{(q)}(m))$ in (#-42) to the class which has the representative of the first Čech cocycle in $C^1(\{U_a\}, N^\vee(m) \otimes \Omega_P^q)$ coming from $C^1(\{U_a\}, I_X(m) \otimes \Omega_P^q)$ as follows.

$$(\#\text{-}48)\qquad \rho^{(q,m)}(\psi) = \left[\{(U_a \cap U_b,\ \omega_{a,b})\}\right], \qquad \left\{(U_a \cap U_b,\ \omega_{a,b})\right\} \in C^1(\{U_a\}, I_X(m) \otimes \Omega_P^q)$$

$$\omega_{a,b} = \frac{(-1)^q}{(q-1)!} \sum_{j_1, j_2, \cdots, j_{q-1}}^{(a)} \left\{ \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} c^{t(q-1),\cdots,t(1)} \right.$$

$$\det \begin{bmatrix} \left(\widetilde{\delta_{t(1)}}\ \widetilde{\mu_{t(1)}^{(a)}}\right) & \left(\delta_{t(2),t(1)}^{(q-1)}\ \mu_{t(2),t(1)}^{(q-1,a)}\right) & \cdots & \left(\delta_{1,t(q-1)}^{(1)}\ \mu_{1,t(q-1)}^{(1,a)}\right) \\ \frac{\partial\ \widetilde{\mu_{t(1)}^{(a)}}}{\partial(Z_{j_1}/Z_a)} & \frac{\partial\ \mu_{t(2),t(1)}^{(q-1,a)}}{\partial(Z_{j_1}/Z_a)} & \cdots & \frac{\partial\ \mu_{1,t(q-1)}^{(1,a)}}{\partial(Z_{j_1}/Z_a)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\ \widetilde{\mu_{t(1)}^{(a)}}}{\partial(Z_{j_{q-1}}/Z_a)} & \frac{\partial\ \mu_{t(2),t(1)}^{(q-1,a)}}{\partial(Z_{j_{q-1}}/Z_a)} & \cdots & \frac{\partial\ \mu_{1,t(q-1)}^{(1,a)}}{\partial(Z_{j_{q-1}}/Z_a)} \end{bmatrix}$$

$$\left. \right\} \left(\frac{Z_b}{Z_a}\right)^{-1} d\left(\frac{Z_b}{Z_a}\right) \wedge d\left(\frac{Z_{j_1}}{Z_a}\right) \wedge \cdots \wedge d\left(\frac{Z_{j_{q-1}}}{Z_a}\right) \otimes Z_a^m$$

Remark 2.2 *Let us return to the setting (#-36). In case of $a_0 = \psi$, it is very difficult to get a "canonical" representative of $\rho^{(q,m)}(\psi)$ in Čech 1-cocycles directly from the element $a_q \in H^0(X, \Omega_P^q(m)|_X)$. Usually, the calculation of $\overline{\delta_{LFT}}(a_q)$ includes a serious ambiguity coming from $C^0(\{U_a\}, N^\vee(m) \otimes \Omega_P^q)$, which makes the infinitesimal lifting problems into a great labyrinth of infinitly many higher obstructions.*

Proof of Main Theorem. To avoid the difficulty described in Remark 2.2, let us recall several results from [14] on the Ogus complex. By [14], we already know that taking the wedges of the short exact sequence:

(#-49) $$0 \longrightarrow \Omega^1_{W/B}(m) \xrightarrow[\alpha_{1-EU}]{} \Sigma^1_{H/W/B}(m) \xrightarrow[\beta_{1-EU}]{} H^m \longrightarrow 0,$$

we can decompose the Koszul-Ogus complex $(\Sigma^\bullet_{H/W/B}(m), \kappa)$ into the short exact sequences:

(#-50)
$$\begin{tikzcd}
 & \Sigma^{q+1}_{H/W/B}(m) \arrow[d, "\beta_{(q+1)-EU}"'] \arrow[dr, "\kappa^{(q+1)}"] & & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^q_{W/B}(m) \arrow[r, "\alpha_{q-EU}"'] \arrow[d] & \Sigma^q_{H/W/B}(m) \arrow[r, "\beta_{q-EU}"] \arrow[dr, "\kappa^{(q)}"'] & \Omega^{q-1}_{W/B}(m) \arrow[r] \arrow[d, "\alpha_{(q-1)-EU}"] & 0, \\
 & 0 & & \Sigma^{q-1}_{H/W/B}(m). &
\end{tikzcd}$$

where the middle horizontal sequence in (#-50) coincides with the q-th wedge of the short exact sequence (#-49). As we know by Lemma 1.14 of [14], putting $\gamma_{q-EU} = \beta_{(q+1)-EU} \circ \nabla^{(q)}_{OG}$, we see :

(#-51) $$\gamma_{q-EU} \circ \alpha_{q-EU} = m \cdot I_d : \Omega^q_{W/B}(m) \longrightarrow \Omega^q_{W/B}(m),$$

and that the homomorphism of abelian sheaves γ_{q-EU} is a differential operator of order 1. Using this homomorphism γ_{q-EU}, we build up an exact commutative diagram by the the middle horizontal sequence in (#-50) tensored with the sheaf of defining ideals $I_{X/W}$ of a closed subscheme X in W and the two horizontal short exact sequences in (#-9) replacing P by W/B.

(#-52)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{X/W}\otimes\Omega^q_{W/B}(m) & \xrightarrow{\alpha_{EN}} & I_{X/W}\otimes\Sigma^q_{H/W/B}(m) & \xrightarrow{\beta_{EN}} & I_{X/W}\otimes\Omega^{q-1}_{W/B}(m) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\times m)} & & \downarrow{\scriptstyle \gamma_{q-EU}} & & \downarrow{\scriptstyle -d_I} & & \\
0 & \longrightarrow & I_{X/W}\otimes\Omega^q_{W/B}(m) & \xrightarrow[\alpha_{LFT}]{} & \Omega^q_{W/B}(m) & \xrightarrow[\beta_{LFT}]{} & \Omega^q_{W/B}(m)|_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r} & & \downarrow & & \| & & \\
0 & \longrightarrow & N^\vee_{X/W}\otimes\Omega^q_{W/B}(m) & \xrightarrow[\overline{\alpha_{LFT}}]{} & \Omega^q_{W/B}(m)|_{X_{(1)}} & \xrightarrow[\overline{\beta_{LFT}}]{} & \Omega^q_{W/B}(m)|_X & \longrightarrow & 0
\end{array}$$

The homomorphisms α_{EN} and β_{EN} above are the restrictions of the homomorphisms α_{q-EU} and β_{q-EU}, respectively. Taking the long cohomology exact sequences of (#-52), we get:

$$(\#\text{-}53)\qquad \overline{\delta_{LFT}}\circ d_I = (-m)\cdot r\circ\delta_{EN} : H^0(W, I_X\otimes\Omega^{q-1}_{W/B}(m)) \longrightarrow H^1(X, N^\vee_{X/W}\otimes\Omega^q_{W/B}(m)).$$

Let us recall that the extension class of the short exact sequence (#-49) with putting $m=0$ is the same class of 1-jet sequence of H, namely the first Chern class $c_1(H)\in H^1(W,\Omega^1_W)$. Using the similar argument in [7], once if we have a global section of $H^0(W, I_X\otimes\Omega^{q-1}_{W/B}(m))$, then, by tensoring the global section, we can compare the top horizontal exact sequence in (#-52) with (#-49) for $m=0$ and see that the map δ_{EN} coincides with the map by coupling the first Chern class $\cup c_1(H)$.

Now we return to the case : $W=P=\mathbb{P}^N(\mathbb{C})$, $H=O_P(1)$, and $B=\mathrm{Spec}(\mathbb{C})$. We can calculate $c_1(O_P(1))\in H^1(P,\Omega^1_P)$ as the extension class of the short exact sequence (#-49) with putting $m=0$ and see that

$$(\#\text{-}54)\qquad c_1(O_P(1)) = \Big[\,\{(U_a\cap U_b,\quad (Z_b/Z_a)^{-1}\cdot d(Z_b/Z_a))\}\,\Big] \in \check{H}^1(\{U_a\},\Omega^1_P).$$

Let us start by putting $a_0=\psi$ in (#-36). Then, applying the equality (#-53),

(#-55)

$$\rho^{(q,m)}(\psi) = \overline{\delta_{LFT}}\circ(\frac{1}{m})d_I\circ\widetilde{\lambda}(\psi) = -r\circ\delta_{EN}\circ\widetilde{\lambda}(\psi) = -\big[c_1(O_P(1))\cup\widetilde{a_{q-1}}\big] = -\big[\widetilde{a_{q-1}}\cup c_1(O_P(1))\big],$$

where $\widetilde{a_{q-1}}\in\Gamma(P, I_X\otimes\Omega^{q-1}_P(m))$ and $\alpha_{(q-1)-EU}(\widetilde{a_{q-1}}) = a_{q-1}\in\Gamma(P,\Sigma^{q-1}_P(m)) = \Gamma(P,\Xi^{q-1,0}_{X/P}(m))$. Essentially, the elements $\widetilde{a_{q-1}}$ and a_{q-1} are the same. However, there is a difference in the efforts to get their explicit local expressions. Thus we use different symbols to distinguish them.

By the equalities (#-54) and (#-55), once we get the local expression of $\widetilde{a_{q-1}}$ on the open set U_a as in Lemma 2.5 in the sequel, we easily get $\omega_{a,b} = -(\widetilde{a_{q-1}}|_{U_a})\wedge((Z_b/Z_a)^{-1}\cdot d(Z_b/Z_a))\in\Gamma(U_a\cap U_b, I_X\otimes\Omega^q_P(m))$ as in the Main Theorem. ∎

In the remaining part of this paper, we calculate the local expression of $\widetilde{a_{q-1}}$ on the open set $U_a = D_+(Z_a)$ by the following three lemmata.

Lemma 2.3 *For an integer $u = 1, \ldots, q-1$ and a section $\psi = \sum_{t(1)=1}^{\sigma(q-1)} \widetilde{M}_{t(1)} e_{t(1)}^{(q-1)} \in \Gamma(P, Z_X^{(q)}(m))$, with putting $a_0 := \psi$ and applying the definition (#-36), we get a global section a_u of $\Xi_{X/P}^{(u,q-u-1)}(m)$:*

(#-56)

$$a_u = (-1)^{\frac{(u+1)u}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(u+1)=1}^{\sigma(q-u-1)} m \cdot c^{t(u),\cdots,t(1)}\, M_{t(u+1),t(u)}^{(q-u)} \eta_{t(u),t(u-1)}^{(q-u+1)} \wedge \eta_{t(u-1),t(u-2)}^{(q-u+2)} \wedge \cdots \wedge \eta_{t(2),t(1)}^{(q-1)} \wedge \widetilde{\eta_{t(1)}} \quad e_{t(u+1)}^{(q-u-1)}.$$

In particular, if $u = q-1$, then $F_0 = S$, which implies $e_{t(q)}^{(0)} = 1 \in S$, $m_{t(q)}^{(0)} = 0$, and $\sigma(0) = 1$, namely $t(q) = 1$, thus we have

(#-57)

$$a_{q-1} = (-1)^{\frac{q(q-1)}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} m \cdot c^{t(q-1),\cdots,t(1)} \left(\mu_{1,t(q-1)}^{(1,a)} \otimes Z_a^{\delta_{1,t(q-1)}^{(1)}}\right) \eta_{t(q-1),t(q-2)}^{(2)} \wedge \eta_{t(q-2),t(q-3)}^{(3)} \wedge \cdots \wedge \eta_{t(2),t(1)}^{(q-1)} \wedge \widetilde{\eta_{t(1)}}.$$

Proof. Apply the induction on the integer u. Since each $\eta_{k,j}^{(i)}$'s and $\widetilde{\eta_u}$'s are ∇_{OG}-closed by the equalities (#-45) and (#-46), we have only to care the term $M_{t(u+1),t(u)}^{(q-u)}$ by Lemma 1.3 to get the section $b_{u+1} = \delta_I^{u,q-u-1}(a_u)$ in the process of carrying out this induction. Recalling the fact that

$$\text{(\#-58)} \qquad \Xi_{X/P}^{u,q-u-1}(m) = \bigoplus_{t(u+1)=1}^{\sigma(q-u-1)} \Sigma_{O_P(1)/P/\mathbb{C}}^{u} \left(m - m_{t(u+1)}^{(q-u-1)}\right) e_{t(u+1)}^{(q-u-1)}$$

to check the coefficients $1/(m - m_{t(u+1)}^{(q-u-1)})$ arising from the revised Ogus derivation $\widehat{\nabla}$, it is easy to see that

(#-59)

$$b_{u+1} = \sum_{t(u+1)=1}^{\sigma(q-u-1)} \left((-1)^{\frac{(u+1)u}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(u)=1}^{\sigma(q-u)} m \cdot c^{t(u+1),\cdots,t(1)} \eta_{t(u+1),t(u)}^{(q-u)} \wedge \eta_{t(u),t(u-1)}^{(q-u+1)} \wedge \cdots \wedge \eta_{t(2),t(1)}^{(q-1)} \wedge \widetilde{\eta_{t(1)}} \right) \quad e_{t(u+1)}^{(q-u-1)}.$$

∎

Lemma 2.4 *On the same assumption of Lemma (2.3), the local expression of the element* $a_{q-1} \in \Gamma(P, \Xi_{X/P}^{q-1,0}(m))$ *on the affine open set* $D_+(Z_a)$ *is given as follows. Here, the mark* $\smile$ *means removing the object on the mark.*

(#-60)

$$a_{q-1}\big|_{D_+(Z_a)} =$$

$$(-1)^{\frac{q(q-1)}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} \sum_{j_1,j_2,\cdots,j_{q-1}}^{(a)} m \cdot c^{t(q-1),\cdots,t(1)} \cdot \mu_{1,t(q-1)}^{(1,a)}$$

$$\frac{\partial\, \mu_{t(q-1),t(q-2)}^{(2,a)}}{\partial(Z_{j_1}/Z_a)} \cdots \frac{\partial\, \mu_{t(2),t(1)}^{(q-1,a)}}{\partial(Z_{j_{q-2}}/Z_a)} \widetilde{\frac{\partial\, \mu_{t(1)}^{(a)}}{\partial(Z_{j_{q-1}}/Z_a)}}\, dZ_{j_1} \wedge \cdots \wedge dZ_{j_{q-1}} \otimes Z_a^{m-q+1}$$

$$+ \quad (-1)^{\frac{q(q-1)}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} \sum_{w=1}^{q-2} (-1)^{w-1} \sum_{j_1,j_2,\cdots,\overset{j_w}{\smile},\cdots j_{q-1}}^{(a)}$$

$$m \cdot c^{t(q-1),\cdots,t(1)} \cdot \mu_{1,t(q-1)}^{(1,a)} \left\{ \delta_{t(q-w),t(q-w-1)}^{(w+1)} \cdot \mu_{t(q-w),t(q-w-1)}^{(w+1,a)} - \sum_{r=0,r\neq a}^{N} \left(\frac{Z_r}{Z_a}\right) \frac{\partial\, \mu_{t(q-w),t(q-w-1)}^{(w+1,a)}}{\partial(Z_r/Z_a)} \right\}$$

$$\frac{\partial\, \mu_{t(q-1),t(q-2)}^{(2,a)}}{\partial(Z_{j_1}/Z_a)} \cdots \overset{w}{\smile} \cdots \frac{\partial\, \mu_{t(2),t(1)}^{(q-1,a)}}{\partial(Z_{j_{q-2}}/Z_a)} \widetilde{\frac{\partial\, \mu_{t(1)}^{(a)}}{\partial(Z_{j_{q-1}}/Z_a)}}\, dZ_a \wedge dZ_{j_1} \wedge \cdots \overset{w}{\smile} \cdots \wedge dZ_{j_{q-1}} \otimes Z_a^{m-q+1}$$

$$+ \quad (-1)^{\frac{q(q-1)}{2}} \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} (-1)^{q-2} \sum_{j_1,j_2,\cdots,j_{q-2}}^{(a)}$$

$$m \cdot c^{t(q-1),\cdots,t(1)} \cdot \mu_{1,t(q-1)}^{(1,a)} \left\{ \widetilde{\delta_{t(1)}} \cdot \widetilde{\mu_{t(1)}^{(a)}} - \sum_{r=0,r\neq a}^{N} \left(\frac{Z_r}{Z_a}\right) \widetilde{\frac{\partial\, \mu_{t(1)}^{(a)}}{\partial(Z_r/Z_a)}} \right\}$$

$$\frac{\partial\, \mu_{t(q-1),t(q-2)}^{(2,a)}}{\partial(Z_{j_1}/Z_a)} \cdots \frac{\partial\, \mu_{t(2),t(1)}^{(q-1,a)}}{\partial(Z_{j_{q-2}}/Z_a)}\, dZ_a \wedge dZ_{j_1} \wedge \cdots \wedge dZ_{j_{q-2}} \otimes Z_a^{m-q+1}$$

Proof. Substitute (#-45) and (#-46) for each $\eta_{k,j}^{(i)}$'s and $\widetilde{\eta_u}$'s in the right handside of the equality (#-57). Then apply the equalities (#-6) and (#-43). ∎

Lemma 2.5 *Under the circumstances, the global section* $\widetilde{a_{q-1}} \in \Gamma(P, I_X \otimes \Omega_P^{q-1}(m))$ *which satisfies* $\alpha_{(q-1)-EU}(\widetilde{a_{q-1}}) = a_{q-1} \in \Gamma(P, \Sigma_P^{q-1}(m)) = \Gamma(P, \Xi_{X/P}^{q-1,0}(m))$ *has the following local expression on the affine open set* $D_+(Z_a)$.

(#-61)

$$\widetilde{a_{q-1}}\big|_{D_+(Z_a)} = \frac{1}{(q-1)!} \sum_{j_1,j_2,\cdots,j_{q-1}}^{(a)} \left\{ \sum_{t(1)=1}^{\sigma(q-1)} \cdots \sum_{t(q-1)=1}^{\sigma(1)} c^{t(q-1),\cdots,t(1)} \right.$$

$$det \begin{bmatrix} \left(\widetilde{\delta_{t(1)}}\ \widetilde{\mu^{(a)}_{t(1)}}\right) & \left(\delta^{(q-1)}_{t(2),t(1)}\ \mu^{(q-1,a)}_{t(2),t(1)}\right) & \cdots & \left(\delta^{(1)}_{1,t(q-1)}\ \mu^{(1,a)}_{1,t(q-1)}\right) \\ \frac{\partial\ \widetilde{\mu^{(a)}_{t(1)}}}{\partial(Z_{j_1}/Z_a)} & \frac{\partial\ \mu^{(q-1,a)}_{t(2),t(1)}}{\partial(Z_{j_1}/Z_a)} & \cdots & \frac{\partial\ \mu^{(1,a)}_{1,t(q-1)}}{\partial(Z_{j_1}/Z_a)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\ \widetilde{\mu^{(a)}_{t(1)}}}{\partial(Z_{j_{q-1}}/Z_a)} & \frac{\partial\ \mu^{(q-1,a)}_{t(2),t(1)}}{\partial(Z_{j_{q-1}}/Z_a)} & \cdots & \frac{\partial\ \mu^{(1,a)}_{1,t(q-1)}}{\partial(Z_{j_{q-1}}/Z_a)} \end{bmatrix}$$

$$\left. \vphantom{\sum} \right\} d\left(\frac{Z_{j_1}}{Z_a}\right) \wedge \cdots \wedge d\left(\frac{Z_{j_{q-1}}}{Z_a}\right) \otimes Z_a^m$$

Proof. By the equality (#-51), we see that $(1/m)\gamma_{(q-1)-EU}(a_{q-1}) = \widetilde{a_{q-1}}$. To get $\widetilde{a_{q-1}}$ from a_{q-1}, it is enough to apply the formula of $\gamma_{(q-1)-EU}$ in Lemma 1.17 of [14]. ∎

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(Received January 9, 2008;Accepted February 5, 2008)